PROCEEDINGS
OF THE
NATIONAL ACADEMY OF SCIENCES
Volume 18 May 15, 1932 Number 5

INTERACTION OF RUFFLED AND ROUNDED GENES OF DROSOPHILA VIRILIS

By G. A. Lebedeff

Department of Genetics, Carnegie Institution of Washington, Cold Spring Harbor, N. Y.

Communicated April 14, 1932

Cases of the interaction of factors, as in the inheritance of a form of comb in poultry, coat color in rodents or fruit color in squashes, are unknown in Drosophila literature. However, there are several cases described of interaction between genes and modifiers, or genes which produce visible effect only in the presence of a particular gene. Bridges and Mohr[1] have shown that the character, vortex, is dependent mainly upon two mutant genes, neither of which alone is sufficient for the production of vortex. Lancefield[2] has described a recessive third chromosome-modifying gene which makes females that are heterozygous for the sex-linked forked gene appear intermediate between forked and wild types. Clausen and Collins[3] have shown that the ski-wing character is due to two independent genes, one dominant and the other recessive. Neither of these genes produces the ski-wing character except when acting together.

In the present paper a case in *Drosophila virilis* is described in which a well-known fifth chromosome recessive gene, ruffled (*ru*)[4] becomes incompletely dominant in the presence of the well-known second chromosome dominant gene for rounded wings (*R*) (unpublished). This change is accompanied by the interaction of *ru* and *R* genes, resulting in the appearance of a new wing character, roofed.

The author wishes to express his gratitude to Doctor M. Demerec of the Carnegie Institution of Washington for helpful suggestions, and to Miss E. deG. McKee for drawings of figure 1.

Origin.—From a cross between a miniature female and a concave telescoped approximated male two cultures (Nos. 2996, 3001) were obtained in which all of the 212 flies were rounded ruffled roofed (Fig. 1*b*). From the subsequent genetical tests it was found that the concave telescoped

approximated flies had also the recessive ruffled (*ru*) gene, and the miniature flies used in this cross carried the dominant gene for rounded wings (*R*). Therefore, the rounded wings in these flies were due to the heterozygous gene *R*, but the appearance of the ruffled, as well as the presence of the newly discovered character, roofed, could not have been accounted for in these F_1 flies on the facts known at that time.

Change in Dominance.—To find out whether or not the interaction between ruffled and rounded genes is responsible for the appearance of ruffled and roofed flies in the F_1 generation, a cross was made between homozygous rounded and ruffled flies. Both parents of this cross were taken from stock cultures kept inbred for many generations. The two cultures (Nos. 4131, 4146) from this cross gave 141 flies, all of which were rounded ruffled roofed.

The result of this cross indicates two interesting facts. The appearance of ruffled flies in F_1 shows that ruffled, which behaved as a simple recessive gene in all previous crosses, becomes dominant in crosses where the rounded gene is involved. On the other hand, the appearance of the entirely new roofed character indicates that this character is a result of the interaction of rounded and ruffled genes.

Second Generation Phenotypes.—The fourteen F_2 cultures obtained from F_1 pairs gave the results summarized in table 1, from which it is evident that five distinct phenotypes appeared in the F_2 generation in a 3:6:3:3:1 ratio. The goodness of fit between the observed and calculated numbers is very good, the *P* being 0.7649.

TABLE 1

SUMMARY OF F_2 DATA

CROSS	NUMBER OF CULTURES	ROUNDED	ROUNDED RUFFLED ROOFED	EXTREMELY ROUNDED RUFFLED ROOFED	WILD TYPE	RUFFLED
Original cross	9	104	250	111	112	32
$R/R \times ru/ru$	5	81	148	77	83	23
Total	14	185	398	188	195	55
Calculated on basis of 3:6:3:3:1 ratio		191.4	382.8	191.4	191.4	63.8

$\chi^2 = 1.8326$, $P = 0.7649$

The 185 flies or $^3/_{16}$ of the F_2 population had rounded straight wings and straight dorso-central bristles with hairs near them. Genotypically they were assumed to be R/R +/+ or $R/+$ +/+.

The 398 flies or $^6/_{16}$ of the population were rounded ruffled roofed, thus resembling their parents. The wings were rounded and curved under; the thorax was in most cases normal; and the tips of the dorso-central

bristles were curled as in homozygous ruffled flies, but the hairs near them were less curled than in homozygous ruffled flies. Their genotype is presumably *R/R ru/+* or *R/+ ru/+* (Fig. 1*b*).

The 188 flies or $^{3}/_{16}$ of the population were rounded ruffled roofed, but all characters were exaggerated (Fig. 1*c*). The short wings were extremely roofed and spread; the thorax was short and broad with two rather distinct humps on it, resembling the vortex character of *Drosophila melanogaster;* the dorso-central bristles and hairs near them were extremely

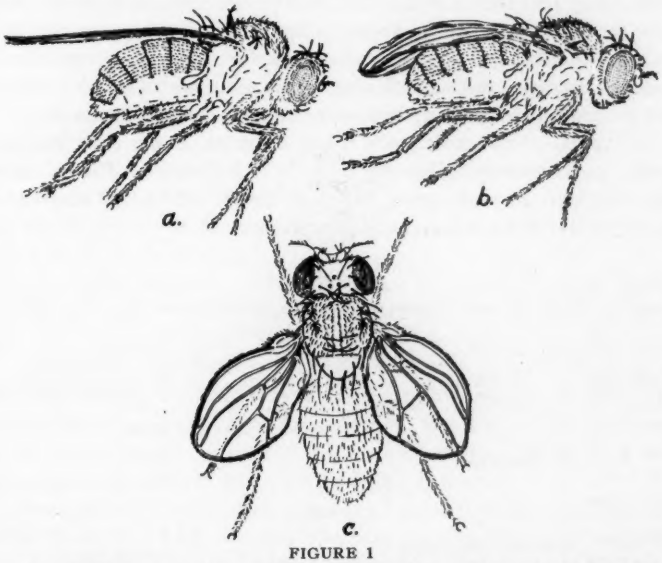

FIGURE 1

(*a*) Ruffled fly (*+/+ ru/ru*); (*b*) Rounded ruffled roofed fly (*R/R ru/+* or *R/+ ru/+*); (*c*) Extremely rounded ruffled roofed fly (*R/R ru/ru* or *R/+ ru/ru*).

curled, more so than in ruffled flies. The genetic constitution of these flies is assumed to be *R/R ru/ru* or *R/+ ru/ru*.

The 195 flies or $^{3}/_{16}$ of the F_2 population were of the wild type, the wings, thorax and dorso-central bristles and hairs near them being normal. Genotypically they were presumably *+/+ ru/+* or *+/+ +/+*.

The 55 flies or $^{1}/_{16}$ of the F_2 population were ruffled (Fig. 1*a*). The tips of their dorso-central bristles and the hairs near them were curled forward toward the midline of the thorax, giving the thorax a ruffled appearance. The genetic constitution was presumably *+/+ ru/ru*.

If the ruffled gene would behave as a simple recessive, the F_2 phenotypical ratio from the cross between rounded (*R R*) and ruffled (*ru ru*)

flies would consist of 9 rounded flies, 3 rounded ruffled, 3 wild type and 1 ruffled (Table 2, column 2). But since the ruffled gene becomes dominant in crosses when the R gene is involved, as was shown by the F_1 result, the F_2 phenotypical ratio is expected to change to 3 rounded, 9 rounded ruffled, 3 wild type and 1 ruffled (Table 2, column 3). Actually the F_2 phenotypical ratio is composed of 3 rounded, 6 rounded ruffled roofed, 3 extremely rounded ruffled roofed, 3 wild type and 1 ruffled (Table 2, column 4).

Comparing columns 2 and 3 of this table we can see that the change of the recessive ruffled gene to a dominant one in the presence of the dominant rounded gene would change the F_2 phenotypical ratio from 9 rounded, 3 rounded ruffled to 3 rounded, 9 rounded ruffled, other phenotypes remaining unchanged, thus not changing the total number of the F_2 phenotypes. Disregarding the roofed character in our data (column 4) rounded and rounded ruffled segregate in a 3:9 ratio. This is another confirmation of our assumption based on the F_1 result, that the recessive ruffled gene becomes dominant in the presence of R.

TABLE 2

F_2 GENERATION PHENOTYPES BASED ON VARIOUS ASSUMPTIONS ABOUT THE BEHAVIOR OF R AND *ru* GENES

FREQUENCY OF F_2 GENOTYPES	R-DOMINANT, *ru*-RECESSIVE	R-DOMINANT, *ru*-DOMINANT WITH R	R-DOMINANT, *ru*-DOMINANT WITH R, FACTORS INTERACTION
1 R/R +/+	9 Rounded	3 Rounded	3 Rounded
2 $R/+$ +/+		9 Rounded ruffled	6 Rounded ruffled roofed
2 R/R +/*ru*			
4 $R/+$ +/*ru*			
1 R/R *ru*/*ru*	3 Rounded ruffled		3 Extremely rounded ruffled roofed
2 $R/+$ *ru*/*ru*			
1 +/+ +/+	3 Wild type	3 Wild type	3 Wild type
2 +/+ +/*ru*			
1 +/+ *ru*/*ru*	1 Ruffled	1 Ruffled	1 Ruffled

Turning back to our actual data we can see that the class of 9 rounded ruffled splits into 6 rounded ruffled roofed and 3 extremely rounded ruffled roofed. This fact merits consideration. The splitting of the 9 class into two classes can be explained by the incomplete dominancy of the *ru* gene in the presence of R, but the appearance of a new roofed character cannot be accounted for by this assumption. The roofed effect is evidently the result of the interaction of R and *ru* genes.

Genetical Test of F_2 Phenotypes.—Four of the five phenotypes observed in the F_2 generation were bred further in order to determine their geno-

types. The results of these tests indicate that our interpretation of their genetic constitution is valid. Three ruffled pairs taken from F_2 cultures gave 122 offspring all of which were ruffled. Crosses of F_2's ruffled flies with wild type flies from the stock gave six F_1 cultures in which every one of the 360 flies was of the wild type. In the F_2 generation in five cultures (Nos. 4370–4374) 228 wild and 69 ruffled flies appeared showing a 3 to 1 segregation. This indicated that ruffled flies are recessive for both genes ($+/+$ *ru/ru*).

From the crosses of extremely rounded ruffled roofed flies with ruffled, both taken from the F_2 generation, there were two cultures from which only extremely rounded ruffled roofed flies were obtained. From the third there were 49 extremely rounded ruffled roofed flies and 41 ruffled. This indicates that extremely rounded ruffled roofed flies are genetypically either R/R *ru/ru* or $R/+$ *ru/ru*.

From the crosses between F_2's rounded and ruffled flies three cultures gave 153 rounded ruffled roofed flies and one culture, 29 rounded ruffled roofed and 22 wild type, which indicates that the genotype of the rounded flies is R/R $+/+$ or $R/+$ $+/+$.

From the cross of wild type flies from the F_2 generation with rounded flies from the stock two cultures were obtained. In the first, all of the 71 flies were rounded; in the second, 31 flies were rounded ruffled roofed and 29 rounded. This indicates that F_2's wild type flies are genotypically $+/+$ $+/+$ or *ru*$/+$ $+/+$. This was further confirmed by the crosses between rounded and wild type flies from F_2. In four cultures from this cross 198 rounded ruffled roofed and 196 rounded flies appeared; in three cultures, 141 rounded and 131 wild type flies appeared.

F_2's rounded ruffled roofed flies have not been tested, but crosses were made of F_1 rounded ruffled roofed flies with rounded flies from the stock. From five cultures 79 rounded ruffled roofed, 83 rounded and 184 wild type flies were obtained, thus giving a 1:1:2 ratio, which shows not only that F_1 rounded ruffled roofed flies are genotypically $R/+$ *ru*$/+$, but also that $R/+$ *ru*$/+$ genotypes are phenotypically ruffled and that $+/+$ *ru*$/+$ phenotypes are wild type. This proves the assumption that the recessive *ru* gene becomes dominant in the presence of R.

Discussion.—Several cases of so-called conditioned dominance have been described. The abnormally bent abdomen of *Drosophila melanogaster* was attributed by Morgan[5] to a gene the expression of which could be controlled by the moisture content of the food. The horned character in Dorset-Down hybrid sheep is dominant in males and recessive in females.

The results of the experiment described in this paper show that the fifth chromosome recessive ruffled gene becomes dominant in the presence of the second chromosome dominant rounded gene. The ruffled gene affects the dorso-central bristles and the hairs surrounding them. It rarely

affects the shape of the thorax but when it does, the thorax becomes slightly telescoped. Ruffled is a well-known recessive gene. When homozygous it has never been observed to overlap the wild type; nor has it been observed to appear in a heterozygous condition. The rounded wing is considered one of the best dominant characters in *Drosophila virilis* since its dominance is complete and the variability slight. But when ruffled is combined with rounded, ruffled becomes dominant, and as such interacts with rounded to cause the appearance of a new character, roofed. Usually in cases of factor interaction the new character which results effaces the expression of the characters involved. From the cross between pea and rose-combed birds, walnut-combed birds appear, neither pea nor rose combs being expressed. In our case, on the contrary, the factors interacting do not lose their identity and the result of the cross between rounded and ruffled is not roofed but rounded ruffled roofed flies. It is true, however, that the ruffled and rounded characters are modified when they interact.

TABLE 3

DESCRIPTION OF THE F_2 PHENOTYPES OBTAINED FROM THE CROSS BETWEEN RUFFLED AND ROUNDED FLIES

GENOTYPE	BRISTLES AND HAIRS	THORAX	WINGS
R/R +/+ *R/+* +/+	Wild type	Wild type	Rounded
R/R +/*ru* R/+ +/*ru*	Ruffled bristles, hairs ruffled only in about 50 per cent flies	Wild type or slightly telescoped	Rounded roofed
R/R ru/ru R/*r ru/ru*	Extremely ruffled	Telescoped with vortex protrusions	Extremely rounded roofed
+/+ +/+ +/+ +/*ru*	Wild type	Wild type	Wild type
+/+ *ru/ru*	Ruffled	Wild type or slightly telescoped	Wild type

Detailed studies of the five F_2 phenotypes (Table 3) show that in rounded ruffled roofed flies (*R ru*/+) the wings are rounded as in the flies of a pure rounded stock; also the type of the dorso-central bristles of *R ru*/+ flies are curled as much as in ruffled flies (+/+ *ru/ru*); but the hairs surrounding the dorso-central bristles are curled in about 50 per cent only of rounded ruffled roofed flies as compared with 100 per cent in ruffled flies. In extremely rounded ruffled roofed flies (*R ru/ru*) the wings are spread and appear to be closely trimmed; the thorax is short and broad with rather distinct vortex protrusions; the dorso-central bristles and hairs near them are extremely curled in comparison with ruffled flies (*r/r ru/ru*).

Summary.—The fifth chromosome recessive ruffled (*ru*) gene in *Drosophila virilis* becomes dominant in the presence of the second chromosome dominant gene for rounded wings (*R*).

The change is accompanied by the interaction of these two genes which results in the appearance of a new roofed character in the F_1 and of five phenotypes in the F_2 in a 3:6:3:3:1 ratio.

[1] Bridges, G. B., and Mohr, O. L., *Genetics*, **4**, 283–306 (1919).

[2] Lancefield, D. E., *Amer. Nat.*, **52**, 462–464 (1918).

[3] Clausen, R. E., and Collins, J. L., *Genetics*, **7**, 385–426 (1922)

[4] Metz, C. W., Moses, M. S., and Mason, E. D., *Carnegie Institution of Washington Pub.* No. 328, 1–94 (1923).

[5] Morgan, T. H., Bridges, C. B., and Sturtevant, A. H., *Bibliographia Genetica*, **11**, 1–262 (1925).

THE MATURATION DIVISIONS AND FERTILIZATION IN EGGS OF SCIARA COPROPHILA: LINT[1]

BY M. LOUISE SCHMUCK AND CHAS. W. METZ

DEPARTMENT OF EMBRYOLOGY, CARNEGIE INSTITUTION OF WASHINGTON, AND DEPARTMENT OF ZOÖLOGY, JOHNS HOPKINS UNIVERSITY

Communicated April 14, 1932

The solution of several problems arising from peculiarities in chromosome behavior in Sciara has been dependent on a study of the maturation divisions in the eggs of these flies. Such a study has been delayed for some time by technical difficulties, but has recently been made possible by the development of a technique which permits study of the chromosomes in entire eggs.[2] Using this technique we have been able to follow the history of the chromosomes during most of the important stages in the maturation divisions, and to trace the sperm during much of its progress in the egg up to the union of the pronuclei. The results are reviewed briefly here in relation to the problems referred to above.

The main questions of interest arise from the fact that in Sciara the sperms regularly receive, and presumably transmit, more than the haploid number of chromosomes. In view of this fact, it is necessary to postulate the occurrence of a compensating process of chromosome diminution at some stage in the life cycle, to restore the chromosome balance. One obvious possibility is that a compensating process may occur during oögenesis which serves to restore the chromosome balance immediately at fertilization. We have examined the maturation divisions in the egg with this possibility especially in mind, but, as noted below, the evidence shows no indication of such regulatory process. It points, rather, to the conclusion that regulation of chromosome number is brought about through

an elimination of chromosomes after fertilization, and that the elimination is correlated with, or similar to, that which occurs during the differentiation of soma from germ-line in these flies.[3]

In the present account we are concerned only with conditions in the germ-line. Those in the soma are considered in earlier papers[3] and in the accompanying paper by Dr. DuBois. Figures of the chromosome groups in this material are shown in the accompanying paper by Dr. DuBois, and in these PROCEEDINGS, **17**, 274 (1931).

Before considering the processes of oögenesis it is necessary to recall more specifically the peculiarities in spermatogenesis, just mentioned, and also to note certain normal variations in chromosome number in this material. The peculiarities of spermatogenesis are as follows: (1) Segregation is highly selective, instead of random; (2) both maturation divisions are asymmetrical, not only as regards cell division, but also as regards chromosome distribution; and (3) each primary spermatocyte gives rise to only one sperm. As a result of the asymmetrical chromosome distribution each sperm regularly receives one or two extra chromosomes. One of these is a rod-like member derived from the "precocious" chromosome which, in the second spermatocyte division, contributes both its daughter halves to the sperm.[4]

The other extra chromosome, if present, is one of the large, V-shaped, "limited" chromosomes (limited to the germ-line).[4] The number of "limited" chromosomes may vary from individual to individual. Typically it appears to be two—in which case the diploid number for the germ-line is ten. Frequently, however, only one is present and the diploid number is nine. Since this variation in number applies to the female, as well as male, it results in correspondingly different numbers of chromosomes in the oöcytes, as noted below.

In the male the "limited" chromosomes do not undergo reduction; hence each sperm receives the full number present in the spermatogonia and different males differ in the number they transmit. These differences in number, however, regardless of sex, have no observable effect on the appearance or genetic behavior of the flies.

The Oöcyte Divisions.—The following account is incomplete as regards certain stages which are passed through very rapidly and which, for this reason, we have not yet been able to find, although many preparations have been made specifically for this purpose. It is expected that these gaps will be filled in a subsequent, more detailed report.

As intimated above, the present observations have been made on eggs prepared *in toto*, with no mechanical injury. All mitotic figures, therefore, are entire and relatively undisturbed. Comparison has been made, however, with sectioned material prepared in this laboratory by Mr. Adrian Ter Louw, in connection with another study, and kindly turned

over to us. Preliminary observations indicating the general nature of the first oöcyte division, as recorded here, were first made by Mr. Ter Louw.

In contrast to the spermatocyte divisions, with their numerous peculiarities, the oöcyte divisions appear to conform to the typical, classical scheme in all essential respects. The metaphase plate of the first oöcyte division includes five chromosomes of the expected types (one large V, one smaller V and three rods). Subsequent events show that these chromosomes are tetrads and that each goes through two maturation divisions in an apparently typical manner (exception being made of cases in which only one large V, or "limited" chromosome, is present—in such cases this chromosome is a dyad and behaves accordingly). The first maturation division is reductional in appearance, the secondary split being inconspicuous. When the large V ("limited" chromosome) is a dyad it passes to one pole undivided, apparently at random.

In the second oöcyte division the dyads divide into their single components and presumably these go to opposite poles in typical fashion, although the second anaphase has not yet been satisfactorily observed.

The above evidence all points to the conclusion that nothing occurs during the maturation of the oöcyte to compensate for the peculiarities of the spermatocyte divisions and the resultant excessive chromosome number in the sperms. Such a conclusion is also supported by the accompanying observations of Dr. DuBois showing the presence of more than the true diploid number of chromosomes in the early cleavage nuclei. It is necessary, therefore, to assume that the excess chromosomes are eliminated, and that probably this elimination occurs during cleavage in conjunction with the elimination which occurs during differentiation of soma from germ-line. In the case of the somatic nuclei it is clear that the elimination occurs at this time, as shown in the accompanying paper by Dr. DuBois; but the elimination from cells of the germ-line has not yet been observed.

From the standpoint of chromosome behavior considerable interest attaches to the mechanism responsible for this postulated elimination from cells of the germ-line. This feature will be discussed more fully elsewhere, but certain aspects should be noted here.

If the egg transmits one, and the sperm two, of the large V-shaped "limited" chromosomes, and one of these is subsequently eliminated, what principle of selection is used in the elimination? A similar question may be asked in the case of the rod-like "precocious" chromosome, for two apparently identical sister chromosomes of this type come from the sperm and one homologous to these from the egg. In both of these cases it seems necessary to assume that either the maternal derivative is regularly eliminated, or else one of the two apparently identical paternal derivatives is regularly selected for elimination. The alternative would be a random

process, which, in our opinion, is still more difficult to conceive. Evidence will be presented elsewhere which points strongly toward the conclusion that, in the case of the rod-like members, one of the two sister (and presumably identical) chromosomes introduced by the sperm is regularly eliminated. It seems probable that this chromosome and the phenomena with which it is involved, are concerned with sex determination (see, e.g., Metz, *Amer. Nat.*, **64**, 380 (1930)).

History of the Sperm.—In contrast to conditions commonly found in other insects, including Diptera, it seems clear that typically only one sperm enters the Sciara egg. This fact is recorded because of its possible bearing on the problem of sex determination here, particularly as regards production of gynandromorphs and of the "bisexual" females in monogenic strains of this species.[5]

So far as observed, other phenomena of fertilization follow the typical scheme. The sperm enters the egg, presumably through the micropyle, during metaphase of the first oöcyte division, and progresses toward the egg nucleus during the progress of the two maturation divisions.

[1] This investigation has been aided by a grant from the NATIONAL RESEARCH COUNCIL, Committee for Research on Problems of Sex.

[2] Schmuck and Metz, *Science*, **74**, 600 (1931).

[3] See accompanying paper by Dr. DuBois; also, Metz, *Biol. Zentralb.*, **51**, 121 (1931); and Metz and Schmuck, these PROCEEDINGS, **17**, 272 (1931).

[4] Metz, Moses and Hoppe, *Zeits. ind. Abst- Vererb.*, **42**, 237 (1926).

[5] Metz, *Quart. Rev. Biol.*, **6**, 306 (1931).

ELIMINATION OF CHROMOSOMES DURING CLEAVAGE IN THE EGGS OF SCIARA (DIPTERA)[1]

BY ANNE MARIE DUBOIS

DEPARTMENT OF EMBRYOLOGY, CARNEGIE INSTITUTION OF WASHINGTON, BALTIMORE, MD.

Communicated April 14, 1932

Sciara coprophila shows the remarkable fact that the number of chromosomes in the cells of the germ line (gonads) is different from that in the cells of the somatic line (all the other organs), as Metz and his co-workers have established.[7-11] The chromosome number is 7 in the male somatic line (Fig. 1, I). This group includes one pair of V-shaped chromosomes, two pairs of rods and one single rod, this last being probably the X-chromosome; consequently, the male soma has the formula XO. In the cells of the female somatic tissues the chromosome number is 8 (Fig. 1, II)—one V-shaped pair and three pairs of rods, one of these pairs being the XX chromosomes. The spermatogonia and ovogonia show exactly the same

number of chromosomes (Fig. 1, III); the group here includes 8 "ordinary" chromosomes (the same set as that in the female somatic cells) and, in addition, one or two so-called "limited" chromosomes; the latter are very large, V-shaped and notably thicker than the others.

It is of interest to determine how such a difference between the chromosome number of soma and germ-line arises during the embryogeny. For this purpose it is necessary to know the set of chromosomes in the fecundated egg and to follow the successive divisions of this primordial nucleus during cleavage. The ovogenesis is quite regular (see accompanying paper by Schmuck and Metz) and produces ovocytes containing 4 "ordinary" chromosomes and one or no "limited" chromosome. On the contrary, spermatogenesis is aberrant[7] and forms only one kind of spermatocyte. This contains 5 "ordinary" chromosomes (the fifth one being a

FIGURE 1

Diagrams showing the chromosome groups in the somatic and germ-lines in both sexes of Sciara coprophila. I, chromosome group in a male somatic cell. II, chromosome group in a female somatic cell. III, chromosome group in the germ cells of both sexes.

FIGURE 2

Diagram of hypothetical chromosome group of fertilized egg.

supplementary rod, proceeding from the division of one of the rods, probably the sex chromosomes, during the second division of maturation) and in addition, one or two "limited" chromosomes (their number not being reduced at the first division of maturation).

All the spermatozoa being cytologically alike, only one type of fecundation is possible and, therefore, all the fecundated eggs, those which shall give female embryos as well as those which shall give male embryos, have the same number of chromosomes (Fig. 2), i.e., 9 "ordinary" ones, the female somatic set and the supplementary rod, and one or two, sometimes three, "limited" chromosomes. (In all the eggs studied I have found 10 or 11 chromosomes, but theoretically a set of 12 may also appear.) The fact that the male and female embryos start their development with the same number of chromosomes is very interesting as regards the question of sex determination in Sciara and shall be discussed in another paper.[5]

The next problem is to detect how the 10 or 11 chromosomes of the egg

nucleus is reduced to 7 and 8, respectively, in the male and female somatic lines. The somatic nuclei should eliminate the "limited" chromosomes and one rod in the female, and the "limited" chromosomes and two rods in the male (compare Figs. 1 and 2). Since it is a large amount of chromatin which must be eliminated, special processes are required in some of the subsequent divisions. The cells of the germ-line should only eliminate one rod in both sexes (excepting cases in which three "limited" chromosomes may be present).

The fecundated egg just after it has been laid goes quickly through the first cleavages. The 4 first cleavages are typical; all the nuclei divide synchronously and in each equatorial plate it is possible to count the full number of 10 or 11 chromosomes. (In some eggs all the nuclei have 10 chromosomes, in others all nuclei have 11 chromosomes.) During these first cleavages the chromosomes are very different in form and size from what they are later in the embryo. They are very long, flexuous and thin; and it is difficult to identify them individually. The "limited" chromosomes are often not distinguishable from the other ones, and the formed equatorial plates are never flat. Later on, at each cleavage, the chromosomes become progressively shorter and thicker and at the time of the ninth cleavage they have their definitive shape.

The fifth cleavage marks the first chromosome elimination. In the late anaphase stage it is easy to see that two chromosomes stay in the middle of the spindle. (Up to the present I have been unable to decide whether these two rod-like chromosomes are really two distinct chromosomes or are products of a late and slow division of a single rod.) When the daughter nuclei enter the resting period the eliminated chromosomes are still to be seen between the nuclei; they swell up and stay in the yolk as little dark-stained clumps of chromatin, which slowly disappear. The fifth cleavage, in the embryology of Sciara, is an important one, for it is at this time that the two first germ cells differentiate and reach the posterior end of the egg, where they will subsequently divide very actively (no longer synchronously with the somatic cells). I have never found an elimination of chromosomes during the sixth cleavage, but frequently in the seventh, eighth and ninth. In some cases all the supplementary chromosomes are expulsed together in one of these cleavages, but more often they are eliminated in two successive cleavages. I suppose that the chromosomes eliminated at this time are the large "limited" ones; they may get broken and the pieces be eliminated in two successive cleavages. This supposition is founded on the fact that often in these cleavages the eliminated chromosomes have not the form of a rod or a V, but are irregular. Some anaphase figures show two rod-like chromatic masses left in the middle of the spindle, some others a little curl of chromatin, and still others only a very long rod.

The only point which I wish to emphasize in this preliminary paper is the fact that in Sciara during the first cleavages an elimination of *whole* chromosomes occurs. So far as I know, this is the first time that such an elimination of many complete chromosomes has been demonstrated.[1,2] Boveri (1887)[2] described for the first time a phenomenon of elimination of chromatic material in Ascaris megalocephala. In this worm, during the first cleavages, the chromosomes in the future somatic cells are broken in pieces and some of the latter are expulsed. This phenomenon of chromatin elimination was later found in some other Ascariidae and in a few insects (Lepidoptera, Coloptera, Diptera; for general review see Fogg, 1930).[6] In the insects the elimination was found during the first cleavages, or during the divisions of maturation, but it was always only an excess of chromatic material which is expulsed and never recognizable, whole chromosomes.

The cause of this chromatin elimination is not yet known. Boveri[3,4] after his classic studies on dispermic eggs and centrifuged monospermic eggs, arrives at the conclusion that this process is not an autonomous act of the chromosomes, but is the result of preëxisting conditions in the cytoplasm of the future somatic cells. In the case of Sciara it is premature to try to explain the cause of the chromosome elimination. One fact is certain; the chromosomes which are eliminated are not eliminated as dead material. In many of my preparations it is evident that the chromosomes have begun to split. Sometimes the two daughter chromosomes are almost separated, and they are always distinctly attached to a spindle fibre. It seems that the process of division in those chromosomes destined to be eliminated is going somewhat more slowly than normally. The division seems to be impeded by some change in the relative viscosities of the cytoplasm and the chromosomes.

[1] This investigation has been aided by a grant from the NATIONAL RESEARCH COUNCIL, Committee for Research on Problems of Sex, grant administered by C. W. Metz.

[2] Boveri, Th., "Über die Befrüchtung der Eier von Ascaris megalocephala," *Sitz. Ber. Ges. Morph. Physiol*, Bd. 3 (1887).

[3] Boveri, Th., "Die Blastodermkerne von Ascaris megalocephala und die Theorie der Chromosomindividualität," *Arch. Zellforsch*, Bd. 3 (1909).

[4] Boveri, Th., "Über die Teilung centrifugierter Eier von Ascaris megalocephala," *Arch. Entw. Mech.*, Bd. 30 (1910).

[5] DuBois, A. M., "Contribution to the Embryology of Sciara Coprophila." In press. (1932.)

[6] Fogg, L. C., "A Study in Chromatin Diminution in Ascaris and Ephestia," J. *Morphol. Physiol.*, **50** (1930).

[7] Metz, C. W., "Chromosomes Studies on Sciara. I. Differences between the Chromosomes of the Two Sexes," *Am. Nat.*, **60** (1926).

[8] Metz, C. W., "Chromosomal Differences between Germ Cells and Soma in Sciara," *Biol. Centralbl.*, Bd. 51 (1931).

[9] Metz, C. W., Moses, M., and Hoppe, E., "Chromosome Behavior and Genetic

Behavior in Sciara. I. Chromosome Behavior in the Spermatocyte Divisions," *Zeit. ind. Abst. Vererb.*, Bd. 42 (1926).

[10] Metz, C. W., and Schmuck, L., "Studies on Sex Determination and the Sex Chromosomes in Sciara," *Genetics*, **16** (1931).

[11] Metz, C. W., and Schmuck, L., "Differences between Chromosome Groups of Soma and Germ-Line in Sciara," *Proc. Nat. Acad. Sci.*, **17**, 272 (1931).

[12] Acknowledgment should be made here of the fact that the first actual observation of chromosome elimination in Sciara was made by Mr. Adrian Ter Louw, in this laboratory, during the course of another study. His evidence indicated the elimination of one chromosome at the eighth or ninth cleavage.

XENIA AND METAXENIA IN APPLES. II

By B. R. Nebel and Iris J. Trump

New York Experiment Station, Geneva, N. Y.

Communicated April 4, 1932

In a previous paper[4] it was shown "that two apples on the same tree originating from different kinds of equally efficient pollen are not strictly alike. The dissimilarities were found in several characters of the apple, such as height, breadth, color, weight and seed length." The experiments were carried out in 1930 on a *Fameuse* tree using *McIntosh* and *Yellow Bellflower* pollen. The data obtained at the time were not considered final. The results of the present paper are likewise only a preliminary contribution toward a problem which may be of greater practical consequence than is now generally realized.

Since the fall of 1930 various papers concerning metaxenia have appeared: Anonymous,[1] Evreinoff,[2] Harrison,[3] Tatarincev,[5] Tschermak[6] and Yasuda.[7] These papers indicate that the phenomenon of metaxenia is becoming recognized.

Procedure.—In the spring of 1931 one to three lateral flowers in 200 clusters on a *McIntosh* tree in the variety collection of the New York State Agricultural Experiment Station were emasculated and bagged. Two days later half the number of clusters were fertilized with *Yellow Bellflower* pollen; the other half with pollen of *Red Astrachan*. Especial care was taken to distribute the two kinds of pollen over the emasculated clusters evenly, the two workers exchanging the two varieties of pollen after every ten pollinations. In the fall 64 apples were harvested from the cross *McIntosh* × *Yellow Bellflower* and 67 from the cross *McIntosh* × *Red Astrachan*.

The apples were arranged in two groups corresponding to the different pollen parents, and roughly graded for size. Every apple, not including

the rotted specimens which were used for seed measurements only, was then given a number which was painted on its surface with India ink. The apples were first measured and weighed, then cut and the seeds removed. The seeds from each apple were placed in a separate paper bag and later measured and weighed. Finally each main group of apples was divided into five subgroups which furnished the samples used for making the chemical tests. The samples were obtained by grating the apples on a kitchen grater. The juice was then roughly separated from the pulp by draining through cheesecloth, hand pressure being applied. The results from the apple and seed measurements and chemical tests are given in table 1.

Results and Discussion.—The measurements of the *height of apples* show that the mean and the standard deviation of height is the same in both groups of apples.

The *breadth of apples* from the cross with *Red Astrachan* pollen is slightly greater. The significance of this difference is only 1.9:1 in terms of odds but the following group of measurements points in the same direction.

The *weight of apples* is significantly larger in the apples from *Red Astrachan* pollen. This difference with a significance of 823:1 is considered an important expression of metaxenia, which is corroborated by the measurements of pH values and acidity by titration.

The *pH values* are significantly lower and less variable for the apple from *Red Astrachan* pollen.

The measurements of the *acidity by titration* differ in the same way, the apples from *Red Astrachan* pollen being more acid.

Differences in *total sugars* could not be demonstrated as several samples were lost in the course of examination.

It may be mentioned that when the two groups of apples were placed on a table side by side the apples from *Red Astrachan* pollen appeared to be the better apples at first glance. It is exceedingly difficult to state accurately which features gave this impression. The measurements indicate that the apples from *Red Astrachan* pollen were superior to those from *Yellow Bellflower* pollen. Higher weight and the presence of more free and titrable acids would make the apples from *Red Astrachan* pollen better keeping and also better looking. Since keeping quality is of extreme economic importance it may well be that the results of this study have a *practical* bearing. The keeping quality could not be examined directly since all available samples were used in the chemical tests.

Observations on the seeds of the apples of the two groups showed the presence of xenia.

Seed length was significantly greater in the apples from *Yellow Bellflower* pollen. The seeds from the same group also seemed slightly more variable in length.

TABLE 1

MEASUREMENTS ON TWO GROUPS OF McINTOSH APPLES FROM POLLINATION WITH YELLOW BELLFLOWER AND RED ASTRACHAN DESIGNATED AS × YB AND × RA

ATTRIBUTES MEASURED	DESIGNATION OF GROUP	UNIT OF OF MEASUREMENT	N*	M* ± PEM*	Dm* ± PEdm	SIGNIFICANCE OF Dm IN TERMS OF ODDS	REMARKS	σ* ± PEσ*	Dσ* ± PEdσ*	SIGNIFICANCE OF Dσ IN TERMS OF ODDS	REMARKS
Height of apples	× YB	mm.	58	51 ± 0.45	0.0 ± 0.6	<1:1		5.04 ± 0.32	0.12 ± 0.45	<1:1	
	× RA		63	51 ± 0.43				5.16 ± 0.31			
Breadth of apples	× YB	mm.	58	61 ± 0.51	1.0 ± 0.7	19:1	Metaxenia	5.72 ± 0.36	0.39 ± 0.51	<1:1	
	× RA		63	62 ± 0.52				6.11 ± 0.37			
Weight of apples	× YB	gram	58	79 ± 2.1	12.0 ± 2.5	828.3:1	Metaxenia	23.2 ± 1.5	1.5 ± 2.0	<1:1	
	× RA		63	91 ± 1.8	.....			21.7 ± 1.3			
Seed length	× YB	1/10 mm.	303	83.5 ± 0.29	2.4 ± 0.39	19300:1	Xenia	7.71 ± 0.21	0.43 ± 0.28	2.21:1	Xenia
	× RA		318	81.1 ± 0.28	.....			7.28 ± 0.19	.....		
Seed length in apples with more than 10 seeds	× YB	1/10 mm	86	87.0 ± 0.43	3.9 ± 0.67	1341:1	Xenia	6.0 ± 0.31	1.6 ± 0.5	31.36:1	Xenia
	× RA		71	83.1 ± 0.51	.....			7.6 ± 0.4	.....		
Averages of seed wt. per apple	× YB	mg.	58	43.3 ± 0.43	0.5 ± 0.62	>1:1		4.8 ± 0.30	0.35 ± 0.42	<1:1	
	× RA		63	43.8 ± 0.44				5.15 ± 0.31			
Averages of wt. of seeds in apples with 12 seeds and more	× YB	mg.	6	39.0 ± 1.05	0.8 ± 1.6	<1:1					
	× RA		8	39.8 ± 1.31							
								3.8 ± 0.74	1.7 ± 1.1	2.21:1	Xenia
								5.5 ± 0.93	.....		
Seed number	× YB	indiv.	64	8.07 ± 0.27	0.17 ± 0.37	<1:1		3.2 ± 0.19	0.08 ± 0.26	<1:1	
	× RA		66	8.24 ± 0.26				3.12 ± 0.18			
pH values	× YB	pH	5	3.43 ± 0.025	0.16 ± 0.04	142.3:1	Metaxenia	0.08 ± 0.02	0.05 ± 0.02	9.9:1	Metaxenia
	× RA		5	3.27 ± 0.008	.....			0.03 ± 0.006	.....		
Acidity by titration	× YB	cc. 1/10 N KOH per 100 cc.	5	42.4 ± 0.28	1.2 ± 0.4	22.24:1	Metaxenia	9.4 ± 2.0	0.0 ± 2.8	<1:1	
	× RA		5	43.6 ± 0.28	.....			9.4 ± 2.0			
Total sugars	× YB	gm. in 100 cc. as invert	2	11.3 ± 0.17	0.45 ± 0.6	<1:1		0.35 ± 0.13	1.01 ± 0.42	?	Number of samples inadequate
	× RA		3	11.75 ± 0.60				1.36 ± 0.38			

*Explanation of signs: N—number of individuals (or samples) measured; M—mean; PEM—probable error of mean; Dm—difference of comparable means; PEdm—probable error of difference of means; σ—standard deviation; PEσ—probable error of standard deviation; Dσ—difference of comparable standard deviations; PEd—probable error of difference of standard deviations.

The *seed length in apples with more than 10 seeds* is considerably greater than the average seed length when all the apples of the experiment are considered. Again the seeds from the *Yellow Bellflower* pollen are longer but this time they are *less* variable than those from the *Red Astrachan* pollen.

Seed weight and *seed number* were not found divergent in the two groups of apples. There was no correlation between seed number and apple size in either of the two groups. If any there may be a slight *negative* correlation between seed number and weight of apples; but this was not investigated.

Summing up the experiences of the senior author with metaxenia in apples it may be stated that:

In 1930 metaxenial effects were found in the crosses *Fameuse* × *McIntosh* and *Fameuse* × *Yellow Bellflower*. In these crosses the metaxenial influence concerned height, breadth and probably weight of apples. Acidity and sugars were not examined.

In 1931 in the crosses *McIntosh* × *Yellow Bellflower* and *McIntosh* × *Red Astrachan* metaxenial effects were recorded concerning the weight of apples as well as their acidity in terms of pH values and titration. These show that the keeping quality is affected in favor of the *Red Astrachan* combination. In 1930 and in 1931 *Yellow Bellflower* pollen produced apples of relatively inferior quality.

Acknowledgment is made to Dr. R. A. Emerson and Professor F. Hartzell for advice and criticism and to Dr. Z. I. Kertesz and Dr. Carl Pederson for their assistance in performing the chemical tests.

[1] Anonymous, "B. P. I. Men Discover a Secret of Pollination," *The Official Record*, **10** (31), 225–226 (1931).

[2] Evreinoff, V., "Sur la question des xenies 2° ordre chez le pecher," *Bull. Mus. Soc. Nat. Hort.*, France, **5** (4), 426–429 (1931).

[3] Harrison, G. J., "Metaxenia in Cotton," *J. Agr. Res.*, **42** (9), 521–544 (1931).

[4] Nebel, B. R., "Xenia and Metaxenia in Apples," *N. Y. State Agr. Exp. Sta. T. B.*, **170**, 1–16 (1930).

[5] Tatarincev, A. S., "The Question About Xenia of the Second Order," *Ann. ldw. Timiriasev. Acad. Moskow.*, **4**, 99–124 (1929).

[6] Tschermak, E., "Über Xenien bei *Leguminosen* Zschr.," *Züchtg.*, **16A**, 73–81 (1931).

[7] Yasuda Sadao, "A Metaxenia-Like Phenomenon Found in Some Plants of *Solanaceae*" Jap., *J. Gen.*, **6** (3–4), 137–142 (1930).

NOTES ON STELLAR STATISTICS. V. ON THE USE OF THE FIRST LAPLACEAN ERROR CURVE

BY WILLEM J. LUYTEN

DEPARTMENT OF ASTRONOMY, UNIVERSITY OF MINNESOTA

Communicated March 21, 1932

The second Laplacean or normal curve of error, more usually known perhaps as the Gaussian curve, came very near, some decades ago, to being regarded as the *ne plus ultra* in discussions of observational errors. This may be due to the sentiment, so admirably expressed by Lippmann in his remark to Poincaré: "...les expérimentateurs s'imaginent que c'est un théorème de mathématiques, et les mathématiciens d'être un fait expérimental." More recently, however, considerable doubt has been thrown upon the practical validity of this normal law, and the feeling has grown that the "law" for the frequency distributions of observational errors can no longer be taken for granted, but that its functional form must be investigated in each individual case.

In a detailed study of an extensive and homogeneous set of data, viz., repeated, identical experiments—the very same data used originally by C. S. Peirce to show that the normal law of error does hold—E. B. Wilson and Margaret Hilferty[1] have shown that these data could not be satisfactorily represented by the normal law.

The similar case of the once paramount method of least squares may also be cited. This is no longer of such universal usage in astronomy since it is often felt that the underlying assumptions are not warranted by the data, and furthermore, because the greater probable accuracy obtained by using it is more than outweighed by the far greater amount of calculation involved as compared to some other simpler and shorter method.

It is a well-known fact, long since recognized by statisticians, that almost any observed frequency curve has longer "tails" and a higher "peak" than the normal law of error would demand. For this reason, frequency curves having less close contact at both ends than the normal curve have sometimes been used to advantage. One of the simplest of these is the first Laplacean error curve, consisting of two simple exponential curves, joining in a sharp peak at the median of the distribution.[2]

During a recent attempt to fit the normal curve to a homogeneous series of observational errors, the writer was struck by the failure of this curve to do so. Accordingly, the first Laplacean curve was tried, with much greater success. On account of the comparatively infrequent use made of this curve thus far, it may be of interest to give the facts of the case in full.

The series of observations chosen consisted of measures of distance of

double stars, published in *Union Observatory Circular No. 78*. The difference between the first and second, the third and fourth, fifth and sixth measures, etc., only was taken; if only three measures were available of one and the same pair, the third has been ignored. These differences were taken in the sense later minus earlier measures, and all possible changes due to orbital motion were ignored. The observations were arranged in class intervals of 0".05 width, the limits of the class intervals being chosen at 0".025, 0".075, 0".125, etc.

The crude values of the first five moments of the resulting frequency curve are given in the second column of table 1. After applying Sheppard's corrections, and referring the moments to the arithmetic mean as origin, the "true" moments are obtained, given in table 1, column 3. Reducing to frequencies proper, the values of column 4 are found.

TABLE 1

ν_0	522	522	1
ν_1	−17	0	0
ν_2	2945	2901	5.56
ν_3	+3031	+3319	+6.35
ν_4	98873	97829	187.2

The arithmetic mean, derived from the first moment, as given in the second column, equals −0.0326 class intervals, or −0".0016, a negligible quantity, in view of the size of the standard deviation equal to $\sqrt{5.56} = 2.32$ or 0".118, giving for the standard deviation of the mean a value of 0".005. This is as it should be, since there is no reason to suppose that the later measures should be systematically smaller than the earlier ones. According to this, the mean error of a single measure is equal to 0".08(3).

If the cumulative frequency curve is plotted on arithmetic probabilty paper, the curve shown in figure 1, no. 1, is obtained. The normal curve of error, if plotted in this way, yields a straight line. It is obviously impossible to draw a straight line through the points of figure 1; hence the observations do not follow the normal law of error. Reading off the curve, there may be obtained:

median −0.6, quartile points −6.2, and +5.3, 16% points,

(which, in a normal curve, would correspond to the standard deviation points) −9.5, +9.5. Hence, the quartile deviation is 5.7, corresponding to a standard deviation of 8.6, while the standard deviation read off the curve is 9.5, and the one actually computed 11.8 (all in class intervals of 0".05 as unit). The S-shape of the curve could have told us this immediately: the central part corresponds to a much smaller dispersion than the tails. If the sign of the observed differences is neglected, and the frequency curve *a priori* considered to be symmetrical, the observations

may be used to plot half this symmetrical curve; this has been done at the lower right in figure 1. Again we see that the points cannot possibly be represented by a straight line.

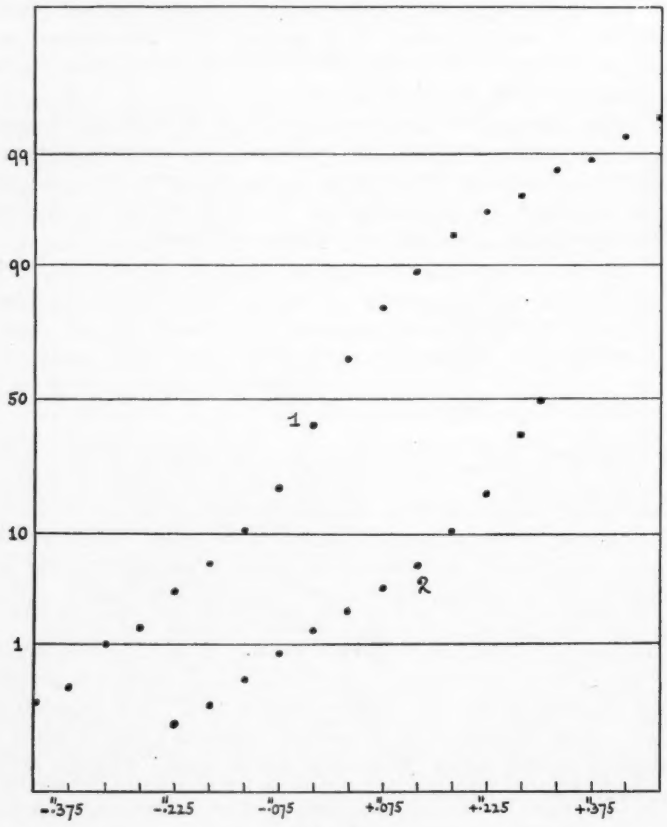

FIGURE 1

Distribution of errors of observation. Ordinates give the percentage number of observations smaller than the value indicated by the abscissae (for set of points indicated by 1). By assuming the frequency curve to be symmetrical and giving frequencies only up to the midway point, the set of points denoted by 2 is obtained; here, however, the zero point of the abscissae has been shifted toward the right by 30".0.

Fitting a Charlier A type curve to the data, we obtain: $A_0 = 1$, $A_1 = A_2 = 0$, $A_3 = -\nu_3/6 = -1.06$, $A_4 = {}^1/_{24}(\nu_4 - 3\sigma^4) = +3.93$. Hence,

$\beta_3 = -0.081$, and $\beta_4 = +0.12$, giving for the skewness and excess the values:

$$S = 3\beta_3 = -0.24 \pm 0.06, \quad E = 3\beta_4 = +0.36 \pm 0.03.$$

The small number of data hardly warrants the use of the fourth moment; the value for the excess, however, is so much larger than its mean error that it cannot but have significance.

In trying to fit a Pearsonian curve, we find for the type criterion κ the value $\kappa = 0.038$, hence a curve of type IV, viz., of the form:

$$y \equiv y_0(1 + x^2/a^2)^{-m} . e^{-\nu \arctan x/a}$$

would do. The fit would be purely formal, and certainly not worth the considerable amount of labor involved in the computation of the constants. The deviation from the normal curve is again glaring, since the values of the Pearsonian constants β_1 and β_2 are found to be $+0.23 \pm 0.05$ and $+6.04 \pm 0.21$, respectively, instead of 0.0 and +3.0 as for the normal curve.

We may now attempt to represent the data by a first Laplacean error curve of the form:

$$y = k/2 . e^{-k|x|} \text{ or } \log y = A + B|x|.$$

The cumulative curve is again a curve of the same type, i.e., a simple exponential curve, which will thus plot as a straight line on arithlog paper. The plot of the observations is given in figure 2, nos. 1 and 2, and it is immediately seen that the representation by a straight line is indeed extremely satisfactory. The asymmetrical character of the curve appears enhanced here, in the fact that it gives two different values for the fundamental constant k on each side of the median. A least-squares solution made from the data by means of the logarithmic form of the first Laplacean curve yields the following two equations for z, the cumulative number of observations, counted from the low end as well as from the high end:

$$\log z = 2.447 + (0.434)(0.64\, x) \text{ (below median)}$$
$$\log z = 2.328 - (0.434)(0.50\, x) \text{ (above median)}$$
(x in class intervals).

If the curve is considered to be symmetrical *a priori*, the two halves may be added, as before, resulting in a plot of points as that given in figure 2, no. 3. A straight line will again fit the data well; the deviations, though decidedly systematic in character, are small enough not to be serious. The value of k calculated for this line is equal to 0.57; with this, the goodness of fit may be computed from Pearson's x^2 test, giving $P = 0.93$, an excellent fit.

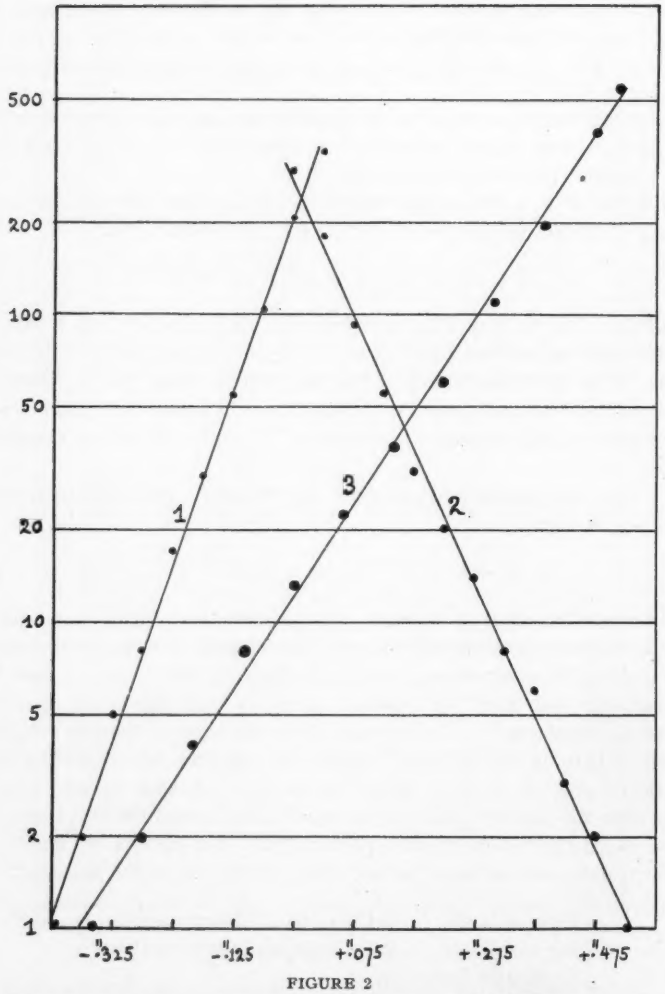

FIGURE 2

Distribution of errors of observation. Line 1: Ordinates indicate the number of observations smaller than the value of the corresponding abscissa. Line 2: Ordinates indicate the number of observations larger than the value of the corresponding abscissa. Line 3: On the supposition that the frequency distribution is symmetrical with respect to the median, the two halves represented by 1 and 2 have been combined into one curve, for which the ordinates represent the number of observations larger in absolute value than a given number; the scale of the abscissae, however, differs from the one used in the other two curves, and is not indicated.

As a corollary to the use of the first Laplacean curve, it may be added that no longer the arithmetic mean and the standard deviation, but the median, and the arithmetic mean error u are the significant constants of the distribution. The value of u is found from k, since $u = 1/k = 1.76$, giving a standard deviation of $u\sqrt{2} = 2.49$, whereas computation from the moments direct gave $\sigma = 2.36$ (class intervals).

It might be added that possibly a dissection of the observed frequency curve into two normal curves, by means of Pearson's nonic might give a good representation, but the minimum number of constants required in that case is three (if the curve is considered symmetrical), as against only one for the first Laplacean.

[1] These PROCEEDINGS, **15**, 120–125 (1929).

[2] Cf. Stromberg, *Mt. Wilson Contrib.* No. 210, 1921; Luyten, these PROCEEDINGS, **9**, 317 (1923); also *Harvard Annals*, **85**, 97 (1923).

THE APPLICATION OF PHOTOELECTRIC CELLS SENSITIVE IN THE INFRA-RED TO STELLAR PHOTOMETRY

BY JOHN S. HALL

YALE UNIVERSITY OBSERVATORY

Communicated March 17, 1932

During the past fifteen months I have been investigating the possibility of using caesium oxide on silver photo-cells in stellar photometry. I have found that these cells may be used to measure very small light intensities with a high degree of precision when sufficiently cooled.

Caesium oxide on silver photo-cells having the two electrodes at the same end were found to have an extremely large leak at room temperature. Dr. C. H. Prescott, Jr., of the Bell Telephone Laboratories in New York, who made the cells I am now using, very kindly made up one with the electrodes coming out at opposite ends. It was noticed that this leak was apparently unchanged but when the polarity was reversed it was very greatly reduced. Dr. Prescott then suggested that perhaps cooling would help decrease the current, which flowed when no visible radiation was incident on the cathode of the photo-cell. We will define this current as "dark current."

I was given the opportunity to follow up this suggestion at the Sloane Physics Laboratory in January, 1931. When the cell was cooled to $-10°$C. the dark current was reduced to less than the sensitivity limit of the galvanometer used. This limit was 3×10^{-11} amperes. Kingsbury and Stilwell, also of the Bell Laboratories (*Phys. Rev.*, **37**, 1549 (1931))

obtained very similar results using a sensitive galvanometer. Koller (*Phys. Rev.*, **33**, 1082(1929)) measured this dark current from caesium oxide on silver in a higher range of temperatures, 100° to 170°C.

A photoelectric photometer was then constructed in such a way that either liquid air or solid carbon dioxide could be used to cool the cell. Complete details of this photometer, together with astronomical data accumulated since last October will be given at a later date. It differs radically from former types in several respects. The photo-cell is made of soda-lime glass instead of fused quartz and is mounted in a thermos bottle. All drying is done quite simply with phosphorus pentoxide. Since the apparatus is attached to the Loomis telescope (a coelostat, the eye end of which is stationary) the task of mounting the apparatus was relatively simple. A specially sensitive Lindemann electrometer is used. It was then found that this dark current could be reduced, in some cases by a factor of more than ten thousand, by cooling the photo-cell with "dry ice." Aside from reducing this dark current to a value which would make stellar measurements possible, cooling stabilizes the photo-current and apparently raises the glow potential. This latter effect has not been thoroughly investigated from fear of damaging a good cell. No appreciable change in sensitivity occurs when the particular cell used is cooled from +5°C. to −40°C. The dark current increases with the voltage impressed on this cell in much the same way as does the photo-current. This would indicate that the two are similar in character. All the cells that I have used, and these include several different makes, reach a steady state in less than a half hour after icing.

None of these cells has a dark current greater than one division in six seconds after it has been cooled to approximately −40° when the cathode potential is a few volts under the glow potential for the cell in question. Since the capacity of the anode-electrometer system was found to be in the neighborhood of 9 cm, and one division per second corresponds to 0.028 volt/sec., this dark current is less than 5×10^{-14} amps. The cell which I am now using, by far the best in this respect, has a dark current which is less than 5×10^{-16} amps. at 135 volts, the highest voltage yet applied to this cell. The important feature of this dark current is that it seems to be very constant as long as the cell is in thermal equilibrium. In the case of the cell mentioned above, higher sensitivity than that obtained with a Lindemann electrometer could undoubtedly be used as far as trouble from the dark current is concerned.

The lens of the Loomis telescope has an aperture of 15 inches. An A0 star of the fourth visual magnitude will give 1 division per second. A fifth magnitude K0 star will give about the same deflection. However, the stability of the apparatus enables one to measure the intensities of fifth magnitude white stars or sixth magnitude red stars with considerable

accuracy. The fact that the atmospheric absorption in the red is much less than that in the blue is obviously also an advantage. Readings taken over a period of a minute have an average deviation of about one second. Somewhat greater sensitivity could undoubtedly be attained by raising the cell voltage. However, until the need arises, this will not be done since it would involve some risk of spoiling an excellent photo-cell.

The point of maximum sensitivity for equal spectral energy is near 8000 Å. After the light from an A0 (white) star has penetrated the atmosphere, been reflected from a mirror and passed through several glass surfaces, about 55 per cent of the total effective energy is of longer wave-length than 6800 Å. For a K0 (yellow) star this ratio is about 75 per cent. When a filter is used which cuts off the energy at approximately 7800 Å, the total effective energy of a white star is reduced by about 80 per cent while that of a red star is reduced by approximately 64 per cent.

Actual observations have shown that the photoelectric current is at least very nearly proportional to the incident light intensity over a range of magnitudes extending from 3.0 to 6.5. The important point at present is that the same relative magnitudes are found for the same stars on different nights to within a high degree of accuracy.

I am very grateful to Dr. Schilt, then at Yale Observatory, now at Columbia University, for his many helpful suggestions and constant encouragement. The coöperation of the Bell Laboratories in general and of Dr. Prescott in particular made this work possible. The G-M Laboratories, Inc., in Chicago and the Westinghouse Laboratory in Pittsburgh also showed noteworthy spirit of coöperation.

THE ROTATIONAL STRUCTURE OF THE ULTRA-VIOLET ABSORPTION BANDS OF FORMALDEHYDE

BY G. H. DIEKE[1] AND G. B. KISTIAKOWSKY[2]

COMMUNICATION NO. 31 FROM THE ALFRED L. LOOMIS LABORATORY IN TUXEDO

Communicated March 28, 1932

We have photographed the ultra-violet absorption bands of formaldehyde in the third order of the 40 ft. spectrograph of the Loomis Laboratory at Tuxedo[3] with a dispersion of about 0.4 Å per mm. The bands are practically completely resolved, and the lines are very sharp. Until now we have measured and studied the bands at 3520, 3430 and 3390 Å. Although not all details of the analysis have been completed yet, we think that the results obtained so far are of sufficient interest to justify their publication. Some of the minor details may require modification

afterward. We are confident, however, that the main results are essentially correct.

Victor Henri, who did most of the pioneer work in this field, and Schou[4] attempted also a rotational analysis of the formaldehyde bands. Our results do not confirm their analysis. This is not surprising, as instruments of much lower resolving power were used by Henri and Schou with which they obtained about 100 lines in every band whereas we were able to measure more than 500 in a single band.

All the bands show quite striking regularities at both ends, which become evident even with a casual inspection. In the center many lines overlap even with our resolution, and accordingly the analysis is much more difficult and uncertain there. It is complicated there, furthermore, by the fact that the lines do not follow any simple formula as they do at the ends of the band. The structure of the bands 3520 and 3390 is identical and can in first approximation be regarded as a $\perp$ band[5] originating from a symmetrical top molecule. This is due to the fact that the moment of inertia about the CO axis is very small and accordingly the two moments of inertia about the axes perpendicular to the CO axis are almost the same. The only bands of this type known so far are some incompletely resolved infra-red bands.[6] The main difference between our bands and those infra-red bands (apart from the fact that they are completely resolved in our case) is the fact that the moments of inertia in the upper and lower states are different which causes successive lines in the various branches to be no more equidistant, and is responsible for the occurrence of typical band heads.

In this communication only those properties of the bands which can be understood with this simple symmetrical top model will be discussed. The deviations which manifest themselves according to theory and observations mainly in the center of the bands will be treated in a subsequent paper.

The rotational energy of a symmetrical top is given by

$$\frac{h^2}{8\pi^2 A} J(J+1) + \frac{h^2}{8\pi^2}\left(\frac{1}{C} - \frac{1}{A}\right)K^2,$$

in which C and A are the moments of inertia about the axis of symmetry and about an axis perpendicular to the axis of symmetry. If we denote, as in diatomic molecules, transitions (in *emission*) in which the total angular momentum J changes by -1, 0, 1 as P-, Q- and R-branches and denote transitions of the component K of the angular momentum about the axis of symmetry by an analogous small letter in front of the P, Q, R we must expect the following branches for a transition in which the electric moment is perpendicular to the axis of symmetry

pP and rR	both strong; they are the only branches for high values of K
pQ and rQ	strong for small values of K
pR and rP	weak and only present for small values of K.

All these branches have been found. They show the following characteristics which fix their nature and allow an unambiguous assignment of quantum numbers.

1. At the violet end of the bands there is a very easily recognizable sequence of about seven R-branches, each of which contains about 15 lines. The spacing of equivalent lines is very nearly the same in all branches and they follow with great precision a Deslandres formula, both the lines in one branch and equivalent lines in the different branches.

2. If all these R-branches are compared with the one lying most to the red, it is seen that in the second one the first line is missing, in the third the first and second lines, etc., until the seventh where the first six lines are missing. This shows that the successive R-branches must be rR branches, i.e., due to successive $K + 1 \longrightarrow K$ transitions. As always $J \geq K$ there must be one more line at the beginning missing for every higher value of K in accordance with what is found.

3. In the red end of the band a few branches can also easily be recognized and shown to be P-branches.

4. If the pP branches belonging to the rR branches mentioned in (1) and (2) are calculated they coincide almost exactly with the empirical P branches mentioned in (3). The number of missing lines increases also here by one if K is increased by one.

5. The correctness of this conclusion is proved by the combination relations which are satisfied within the limits of the errors of measurement. These combination relations allow the separation of the rotational differences of the initial and final electronic state and permit an unambiguous assignment of quantum numbers both for J and K.

6. The value of the moments of inertia calculated from the bands 3520 and 3390 which arise from the transitions from two successive vibrational states ($v = 0$ and $v = 1$) of the upper electronic level to the normal state of the molecule are

		$A \cdot 10^{40}$	$B \cdot 10^{40}$	$C \cdot 10^{40}$
Normal state		24.58	21.63	2.951
Upper state	$v = 0$	27.98	24.81	3.176
	$v = 1$	28.13	24.94	3.190

These values probably will undergo slight modifications, when the influence of the assymmetry of the molecule on the structure of the bands is more rigorously taken into account.

From these values it follows that in the normal state of the formaldehyde molecule the distance between the two hydrogen nuclei is 1.88 Å. If it is assumed that the CH distance is the same as in other molecules, e.g., CH namely, 1.1 Å, the angle between the two CH bonds becomes about 120° and the CO distance 1.21 Å. This value is slightly larger than in the normal CO molecule where it is 1.15 Å. If the angle between the two CH bands is assumed to be the tetrahedron angle (110°) the CH distance is 1.15 Å and the CO distance 1.19 Å. If these values are confirmed by the more rigorous calculations, it would mean that the bonds in formaldehyde are slightly weakened.

7. It appears that the intensity of successive rR-branches is alternately

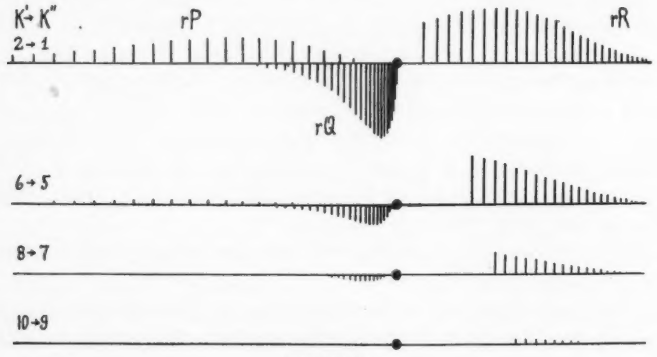

FIGURE 1

Intensities in the absorption bands of formaldehyde at room temperature. The spacing of the lines is chosen arbitrarily so as to bring out the intensity relations most clearly. In the *ultra-violet* absorption bands the lines converge much more strongly, so that the head of the R-branches is reached for low values of J. In the *infra-red* absorption bands the lines converge very little, so that all the lines of the Q-branches practically fall on top of each other. The dots denote the origins of the individual sub-bands.

strong and weak, and that the transitions from an even K of the excited electronic state to an odd K of the lower state are the strong ones. Such behavior must be expected, if the rotation occurs about the CO axis of the CH_2O molecule. Then only the two hydrogen atoms contribute to the moment of inertia and the same kind of intensity alternations must occur as in the hydrogen molecule. The observed intensities are consistent with the expected ratio 3:1.

Successive pP (and also rQ, etc.) branches show the same behavior.

8. In looking for the rQ, pQ, rP and pR branches it is necessary to take into account the intensities of these branches. If the moments of inertia are known, the intensities of all the lines in a band can be cal-

culated. We did this with the values given above and found an excellent agreement with the observed intensities. In figure 1 intensities of the P, Q and R-branches are given for several $K + 1 \longrightarrow K$ transitions. For small values of K the intensity relations are the same as in bands of diatomic molecules (e.g., the $2 \longrightarrow 1$ sub-band has the same intensities as a $\Delta \longrightarrow \Pi$ band of a diatomic molecule). With increasing K the rP-branches become so weak that they are beyond the possibility of observation. The rQ branches also disappear for high values of K and then only the rR-branches are left. For the $K - 1 \longrightarrow K$ transitions analogous relations hold.

These theoretically expected intensity relations are exactly those observed in the bands. In addition to this, as mentioned under 7, all lines coming from a level with an odd value of K are about three times as weak as compared with those coming from a level with an even K.

9. The moments of inertia of the initial state increase with increasing vibrational quantum number. That is also the case for diatomic molecules. This fact is important for the identification of the nuclear vibration. But we prefer postponing the discussion of this point for a subsequent paper. On photographs taken with a quartz spectrograph and long absorption tubes with formaldehyde under relatively high pressure, we found a number of new bands not noticed by previous observers. These seem rather essential for a discussion of the vibrational structure of the band system. But as the nature of the bands can be recognized with much more certainty, if the rotational structure is known, and as we hope to be able to analyze their fine structure this summer, we leave the matter until then.

10. The band at 3430 Å shows in principal the same structure as the two other bands. The moments of inertia for the excited state are $A = 28.16$, $B = 24.99$, $C = 3.173 \cdot 10^{-40}$. The band is a little weaker than the other two and accordingly not so well developed. Also there seem to be quite distinct deviations from the simple quadratic formula which make especially the value of C less accurate than for the other two bands. Nevertheless, we can say that the values for A and B are about the same as those for the initial state of λ 3520 ($v = 0$), whereas the value for C is very nearly the same as that for the initial state of λ 3390 ($v = 1$). In the upper state of λ 3420 another vibration is excited, and it deforms the molecule in a way different from that of the vibration responsible for the initial state of the band at 3390.

11. Henri and Schou gave a partial vibrational analysis of the band system which was corrected by Herzberg.[7] According to Herzberg the lower state of the α band at 3700 is not the normal state of the molecule but has one vibrational quantum. We can confirm this by observations with the gas at 300°C. The α band is very much strengthened then

with respect to the other bands which shows that its lower state must lie somewhat above the ground level of the molecule.

In conclusion we wish to express our thanks to Mr. Alfred L. Loomis for his hospitality and the opportunity to use the facilities of his laboratory. We are also grateful to Mrs. Hilde Kistiakowsky for her help with the wave-length calculations.

[1] Johns Hopkins University.

[2] Harvard University.

[3] A. L. Loomis and G. B. Kistiakowsky, *J. Sci. Instr.*, **4**, 201 (1932).

[4] V. Henri and A. Schou, *Zeit. Physik.*, **49**, 774 (1928).

[5] For nomenclature, etc., see, e.g., D. M. Dennison, *Rev. Modern Phys.*, **3**, 280 (1931).

[6] G. A. Stinchcomb and E. F. Barker, *Phys. Rev.*, **33**, 305 (1929); R. M. Badger and R. Mecke, *Z. physik. Chem.*, **5**, 333 (1929); R. M. Badger, *Phys. Rev.*, **35**, 1038 (1930).

[7] G. Herzberg, *Trans. Faraday Soc.*, **27**, 379 (1931).

THE CILIATION OF THE OVIDUCTS OF REPTILES

BY P. S. CROWELL

ZOÖLOGICAL LABORATORIES, HARVARD UNIVERSITY

Communicated March 30, 1932

An examination of the oviducts of several species of reptiles has been made to determine the way in which these organs act in the transportation of sperms and eggs. In particular the cilia in these tubes have been studied. Parker (1928 a, b; 1930) found in the painted turtle, Chrysemys picta (Schneider), and in the pigeon, Columba livia Gmel., that though the cilia on the greater part of the wall of the oviduct beat toward the cloaca, some beat in the opposite direction. In these animals there is a band of pro-ovarian cilia which extends from the isthmus through the albumen-secreting portion of the duct to the infundibulum. The probable function of these cilia is to carry sperms up the duct to the point where fertilization takes place. I have tried to determine whether a similar condition occurs in other reptiles.

At the outset I examined the oviducts of Chrysemys picta and of the larger terrapin, Pseudemys elegans (Wied). In both of these species the pro-ovarian tract can be found with ease. Hence I can confirm Parker's statement for the condition in Chrysemys and add Pseudemys to the list of animals showing a pro-ovarian tract.

In the "horned toad," Phrynosoma cornutum (Harlan) the length of the albumen-secreting part of the duct is only about two centimeters, or less

than one-fourth of the total length of the oviduct. This is much shorter than the corresponding region in the turtle, where somewhat more than half of the oviduct is albumen secreting. In Phrynosoma, in this short part of the duct, there is a tract of cilia which beat toward the infundibulum. This tract is on that side of the duct which is attached to the supporting membrane. The tract starts a few millimeters from the isthmus and passes through the rest of the albumen-secreting part of the duct to the infundibulum, but it does not continue over this part. The width of the pro-ovarian band is approximately one millimeter, which is about one-fourth of the circumference of the oviduct in this region.

In the fence lizard, Sceloporus undulatus (Latrielle), the length of the albumen-secreting part of the duct is about fifteen millimeters and the circumference of this part of the tube only two millimeters. From the small size of both the oviducts and the ovaries I judge that the specimens under examination were not fully mature. The oviduct may be slit lengthwise and spread out on a slide. The cilia on its exposed inner surface then may be examined under the compound microscope with light transmitted through the wall of the duct. The oviducts of this animal are thus especially favorable for investigation in spite of their small size. In Sceloporus I have found a band of cilia which beat toward the infundibulum. The cilia of this region, in the few cases examined, were far more active than the cilia in the rest of the duct. The position and extent of the pro-ovarian tract corresponds closely with that in Phrynosoma.

From these observations it is clear that the oviducts of the two lizards examined have a pro-ovarian ciliary tract much like that in the turtle, except that it is relatively shorter as is also the whole albumen-secreting part of the lizard oviduct.

Summary.—In the oviducts of the lizards Phrynosoma and Sceloporus there is, in the upper albumen-secreting part, a tract of cilia which beat toward the infundibulum. This appears to be comparable to that described in the turtle and in the pigeon. Its function is probably to convey sperms up the oviduct.

Parker, G. H., "Ciliary Currents in Oviducts of Turtles in Relation to Transportation of Spermatozoa," *Proc. Soc. Exptl. Biol. Med.*, **26**, 52 (1928a).

Parker, G. H., "The Direction of the Ciliary Currents in the Oviducts of Vertebrates," *Amer. Jour. Physiol.*, **87**, 93–96 (1928b).

Parker, G. H., "The Ciliary Systems in the Oviduct of the Pigeon," *Proc. Soc. Exptl. Biol. Med.*, **27**, 704–706 (1930).

THE COMBINATION OF CARBON MONOXIDE AND OXYGEN UNDER THE INFLUENCE OF RADON*

BY S. C. LIND AND CHARLES ROSENBLUM

SCHOOL OF CHEMISTRY, UNIVERSITY OF MINNESOTA

Read before the Academy, Monday, April 25, 1932

The observations of Lind and Bardwell[1] showed that in many cases where an inert gas is present with one or more reactants in a gaseous system exposed to the alpha radiation from radon contained in the mixture, the rate of reaction is proportional to the total ionization produced in the system including that on the inert gas. This raised the question as to why carbon dioxide had not appeared to exert a similar influence on the oxidation of carbon monoxide[2] previously investigated.

In the present work, carbon dioxide has been added to a stoichiometric mixture of carbon monoxide and oxygen, equal to the amount of carbon dioxide that could be finally produced in the completed reaction. The rate of reaction in this mixture was compared experimentally with the rate of a mixture of carbon monoxide and oxygen of the same partial pressures, but without any initial carbon dioxide, when brought about by the alpha radiation from the same amount of radon in both cases.

To anticipate the conclusion of the investigation, it may be stated that carbon dioxide has been found to exert a positive influence on the rate of reaction, but one which is small relative to that which it would have exerted had all of the ions produced on it been equally effective with those produced in the reacting gases. Instead of the ions of carbon dioxide being found 100% efficient in promoting the oxidation of carbon monoxide, the experiments enable the assignment to carbon dioxide of a rather definite efficiency of only 14.5%. This small efficiency explains the failure to detect it in the earlier experiments since there appears to have been a compensation of two oppositely directed influences, the first half of the reaction beginning with abnormally high velocity which diminishes to a minimum about midway in the reaction, from which point a gradual rise takes place that is no doubt due to the accumulation of carbon dioxide in the system. Therefore, the introduction of a large amount of initial carbon dioxide has brought its accelerating influence into greater prominence.

Experimental.—The experiments consisted in following the reactions manometrically in spherical vessels using the general methods that have been employed in the work already cited. The reaction vessels were kept in a thermostat at 25°C. The mercury manometer levels were read with the aid of a cathetometer.

Customary methods were used for the preparation, purification and analysis of the gases. Carbon monoxide was prepared by the action of

formic acid on concentrated sulphuric acid; oxygen, by heating a 3:1 mixture of potassium permanganate and barium peroxide; and carbon dioxide, by heating sodium bicarbonate, from which the 2:1 carbon monoxide–oxygen mixtures were prepared. The stock mixtures were shown by explosion to contain more than 99.5% of the stoichiometric mixtures. The carbon dioxide used was better than 99.6% pure, while the carbon monoxide and oxygen were over 99.7% pure.

TABLE 1

THE CARBON MONOXIDE OXIDATION BY RADON

EXPT. 1. $2CO(g) + 1O_2(g) = 2CO_2(g)$ $D = 1.877$ CM. $E_0 = 50$ MC. RN

P_{total} MM.	P_{2CO+1O_2} MM.	$P^{i}_{CO_2}$ * MM.	$e^{-\lambda t}$
629.7	629.7	0	1.0000
617.9	594.6	23.4	0.9765
608.3	565.8	42.6	0.9524
597.8	534.4	63.5	0.9220
586.9	501.7	85.3	0.8869
573.4	476.4	102.2	0.8431
554.3	404.4	150.2	0.7778
539.0	358.8	180.6	0.7100
522.4	309.3	213.6	0.6242
504.5	256.0	249.1	0.5214
483.6	193.9	290.5	0.3839
471.7	158.7	314.0	0.2929
461.7	129.0	338.8	0.2174
454.2	106.8	348.6	0.1538

EXPT. 2. $2CO(g) + 1O_2(g) + 2CO_2(g) = 4CO_2(g)$ $E_0 = 52.5$ MC. RN $D = 1.904$ CM., INITIAL $P_{CO_2} = 418.7$ MM.

P_{total} MM.	P_{2CO+1O_2} MM.	$P^{i}_{CO_2}$ * MM.	$e^{-\lambda t}$
1035.4	618.2	418.7	1.0000
1021.7	577.1	446.1	0.9759
1010.8	544.4	467.9	0.9512
998.8	508.5	491.8	0.9208
970.2	423.0	548.8	0.8416
948.7	358.8	591.6	0.7750
933.8	314.4	621.2	0.7100
912.4	250.8	663.7	0.6235
894.0	196.1	700.1	0.5207
871.5	129.3	744.6	0.3834
857.8	88.7	771.7	0.2926
843.0	44.9	800.9	0.1534

$$\frac{(E_0)_2}{(E_0)_1} = 1.048$$

* Pressures of CO_2 corrected to ideal gas behavior.

In table 1 are given the results of experiments 1 and 2, the latter being the catalyzed reaction. Here, and in subsequent tables, pressures are calculated for 0°C. Correction is made for the non-ideal behavior of carbon dioxide.

The fact that experiment 2 with initial carbon dioxide present is proceeding faster than experiment 1 can best be shown by plotting the values of log $P_{(2CO+1O_2)}$ as ordinates against $e^{-\lambda t}$ as abscissae in which $e^{-\lambda t}$ is the usual expression for logarithmic decay of radon. The results of this plot are shown in table 2 in the form of a catalytic factor by which is meant the ratio of the velocity of the reaction with initial carbon dioxide $\left(\frac{k\mu}{\lambda}\right)'_C$ to that without $\left(\frac{k\mu}{\lambda}\right)'$. In table 2 the catalytic factor[3] appears as the ratio of the slope of experiment 2 to the slope of experiment 1 after making slight

corrections for the difference in diameter of the reaction spheres[4] and for small differences in the quantities of radon employed.

TABLE 2

CATALYTIC FACTORS

	SLOPES OF LOG P^{2CO+1O_2} VS. $e^{-\lambda t}$ CURVES			
$e^{-\lambda t}$	EXPT. 2	EXPT. 1	SLOPE OF 2 / SLOPE OF 1	CATALYTIC FACTORS
1.0	1.415	1.044	1.355	1.331
0.9	0.974	0.794	1.227	1.205
0.8	0.974	0.794	1.227	1.205
0.7	1.016	0.782	1.300	1.276
0.6	1.058	0.777	1.361	1.336

$$\frac{D_2^2}{D_1^2} \div \frac{(E_0)_2}{(E_0)_1} = 0.892$$

Kinetic Equation for Efficiency Factor.—The general kinetic equation to express the velocity of a reaction proceeding under the influence of radon[5] is

$$\text{const.} = \left(\frac{k\mu}{\lambda}\right)' = \frac{2.303\ \Delta \log P_t}{E_0 \cdot \Delta e^{-\lambda t}}, \qquad (1)$$

in which P_t is the variable partial pressure of the reactant gas at any time t, E_0 is the initial radon in the system and $\left(\frac{k\mu}{\lambda}\right)'$ is a velocity constant which for any given experiment has the dimensions of amount of reaction per 1 curie of radon. In the case of the presence of a foreign gas, the ions of which contribute to the reaction rate with 100% efficiency, Lind and Bardwell[6] developed the equation:

$$\left(\frac{k\mu}{\lambda}\right)' = \frac{2.303\ \Delta \log \left(P_t + \frac{i_c}{i} P_c\right)}{E_0\ \Delta\ e^{-\lambda t}} \qquad (2)$$

in which i_c is the relative specific ionization of the inert (catalytic or sensitizing) gas, i is the specific ionization of the reactant or mixture of reactant gases, and P_c is the partial pressure of the inert gas. The term $\frac{i_c}{i} \cdot P_c$ is the equivalent pressure of reactants in which the same ionization would be produced by radon as in the reactants themselves at this same pressure. Therefore, the sum of the equivalent pressure and actual pressure of reactants is proportional to the ionization that would be produced in this total pressure of reactants under the same conditions, assuming 100% catalytic efficiency of the ions of the chemically inert gas.

In the present work, however, it is found that the efficiency is only x%, and therefore a new expression is used:

$$\left(\frac{k\mu}{\lambda}\right)'_{\text{corr}} = \frac{2.303\,\Delta \log \left(P_t + x\frac{i_c}{i}\cdot P_c\right)}{E_0\,\Delta\, e^{-\lambda t}} \qquad (3)$$

in which the term $x\frac{i_c}{i}\cdot P_c$ gives the reduced ionic equivalent. For the case of carbon dioxide influencing the carbon monoxide oxidation, the value of P_{CO_2} is variable with time since CO_2 is produced by the reaction. To determine the value of x for carbon dioxide by a trial and error method would be extremely laborious and uncertain. However, the definition of the catalytic factor permits a simple experimental determination of the efficiency of carbon dioxide as a catalyst or sensitizer.

After having reduced the velocity constants to a common basis of equal spherical volume in the two cases and furthermore by considering equal time intervals so that the decay constant cancels out, we may write:

$$[\Delta \log P_{(2CO+1O_2)}]_{\text{catalyzed}} = \text{constant} \times [\Delta \log P_{(2CO+1O_2)}]_{\text{normal}} \qquad (4)$$

from which the course of one reaction may be predicted from that of the other if the catalytic factor is known. Since the initial $P_{(2CO+1O_2)}$ is not identical for both reactions of the pair but differs by a few millimeters, we may calculate exactly the pressures in either reaction from the corresponding pressures in the other by applying equation (4). Table 3 shows this calculation. Pressures for the first parts of experiment 2 are computed on the basis of interpolated values of log $P_{(2CO+1O_2)}$ from the normal reaction at rounded values of $e^{-\lambda t}$.

TABLE 3

THE CARBON MONOXIDE OXIDATION AT ROUNDED VALUES OF $e^{-\lambda t}$

			EXPERIMENT 1				EXPERIMENT 2	
$e^{-\lambda t}$	CATALYTIC FACTOR	AVERAGE FACTOR	$P_{(2CO+1O_2)}$	LOG $P_{(2CO+1O_2)}$	Δ LOG $P_{(2CO+1O_2)}$	Δ LOG $P_{(2CO+1O_2)}$	LOG $P_{(2CO+1O_2)}$	$P_{(2CO+1O_2)}$
1.0	1.331		629.7	2.7991			2.7991	629.7
		1.268			0.0882	0.1118		
0.9	1.205		513.9	2.7109			2.6873	486.7
		1.205			0.0790	0.0952		
0.8	1.205		428.5	2.6319			2.5921	390.9
		1.241			0.0788	0.0978		
0.7	1.276		357.4	2.5531			2.4943	312.1
		1.306			0.0773	0.1010		
0.6	1.336		299.1	2.4758			2.3933	247.3

This information may now be used to yield the value of x. To simplify notation, let $P_{(2CO+1O_2)} = P$ and $\frac{x\ i_{CO_2}}{i_{(2CO+1O_2)}}\cdot P_{CO_2} = P_c$. From a consideration of equation (3) we may say from

$$\frac{\left[\dfrac{\Delta \log (P + P_c)}{\Delta e^{-\lambda t}}\right]_{\text{catalyzed}}}{\left[\dfrac{\Delta \log (P + P_c)}{\Delta e^{-\lambda t}}\right]_{\text{normal}}} = 1$$

that for equal time intervals

$$\left[\frac{(P + P_c)_{t_1}}{(P + P_c)_{t_2}}\right]_{\text{catalyzed}} = \left[\frac{(P + P_c)_{t_1}}{(P + P_c)_{t_2}}\right]_{\text{normal}} \qquad (5)$$

Applying equation (5) to the pressure in table 3, quadratic equations may be set up for several intervals and solved for x. These operations are summarized in table 4.

TABLE 4*

CALCULATION OF CATALYTIC EFFICIENCY (X) OF CARBON DIOXIDE

$e^{-\lambda t}$	EXPERIMENT 1			EXPERIMENT 2		
	$P_{(2CO+1O_2)}$	$1.48P_{CO_2}$	$P'_{(2CO+1O_2)}$	$P_{(2CO+1O_2)}$	$1.48P_{CO_2}$	$P'_{(2CO+1O_2)}$
1.0	629.7		629.7	629.7	620.1	629.7 + 620.1x
0.9	513.9	113.9	513.9 + 113.9x	486.7	760.7	486.7 + 760.7x
0.8	428.5	197.9	428.5 + 197.9x	390.0	855.0	390.0 + 855.0x
0.7	357.4	267.9	357.4 + 267.9x	312.1	932.5	312.1 + 932.5x
0.6	299.1	325.3	299.1 + 325.3x	247.3	996.3	247.3 + 996.3x

* Columns headed

$$P'_{(2CO+1O_2)} \text{ equal } P_{(2CO+1O_2)} + x \frac{i_{CO_2}}{i_{(2CO+1O_2)}} \cdot P_{CO_2}$$

$$\text{The ratio } \frac{i_{CO_2}}{i_{(2CO+1O_2)}} = \frac{1.52}{1.03} = 1.48.$$

Substituting values of $P' = (P + P_c)$ in equation (5) for the last three intervals[7] in table 4, three values of x are obtained: 0.142, 0.144 and 0.148, which have an average value of 0.145 or 14.5%.

The constancy of the values of x are indicative of the validity of the above calculation. In subsequent calculations the average value of x = 14.5% will be used to find the true velocity of the alpha-ray reaction between carbon monoxide and oxygen. Though the value was determined for only a portion of the whole course of the reaction, it will be seen in a later section to hold for the whole.

The Recoil Atom Effect.—A study of the chemical effect of recoil atoms which was first discovered by Lind in the case of water synthesis[8] has hitherto been confined to that reaction, in which it is particularly marked on account of the continued removal of the reaction product, water, from the gas phase. It therefore appeared to be of special interest to extend the study of the recoil-atom effect to a reaction where the product remains gaseous and can, therefore, exert a partial "screening" of the recoil atoms. Such a case is presented in the oxidation of carbon monoxide.

If we adopt the method of finding the pressure at which the combined action of alpha-rays and recoil atoms is just twice the rate due to alpha-rays alone, we may employ a modified form of the expression introduced by Lind[9] $\frac{\alpha + R}{\alpha}$ in which α expresses the alpha-ray effect and R the recoil atom effect. By definition, they are equal to each other at the pressure sought, and at any other pressure, alpha varies directly with the pressure and may be placed equal to it at any pressure while R remains constant and equal to alpha at the point of equality. Therefore the expression $\frac{\alpha + R}{\alpha}$ is the abnormality factor which may be used to correct for the recoil atom effect.

In the present case we must substitute for alpha a new value α' which is really made up of the direct alpha-ray effect on the reactants plus that due to the action of alpha-particles on the inert gas carbon dioxide. The recoil atom effect R is that of recoil atoms on the pure reactant after correction has been made for the inert gas. Actually the alpha-ray effect in this case never drops to a point where it becomes equal to the recoil atom effect since it is sustained by the presence of the inert gas.

Table 5 contains the actual data and also the interpolated data in order to furnish more regular intervals. The column $P_{(2CO+1O_2)} = \alpha$ shows the course of the reaction with reference to the drop of partial pressure of the reactant; the column for α' gives the same corrected for the ionization contributed by the carbon dioxide; $\left(\frac{k\mu}{\lambda}\right)'_c$ is the velocity calculated from the values of alpha; $\left(\frac{k\mu}{\lambda}\right)'_{corr.}$ is calculated from the values of α' and therefore represents the velocity corrected for catalytic effect of inert gas. Neither of these approaches constancy.

The last column in which correction is made also for the recoil atom effect represents splendid constancy almost to the end of the reaction thus establishing the validity of the correction $\frac{\alpha' + R}{\alpha'}$, while the factor $\frac{\alpha + R}{\alpha}$ evidently is too large. By observing the values for $\left(\frac{k\mu}{\lambda}\right)'_{corr.}$ it will be seen that after correction for catalytic effect of carbon dioxide, they do exhibit a marked tendency to rise owing to the recoil atom effect, but never attain a value twice the original on account of the screening effect of carbon dioxide. Therefore, the actual value of R cannot be obtained from the data so directly as in the case of the water system and two methods for calculating R have been employed.

One method assumes inverse proportionality of R to diameter and specific ionization. The above-mentioned study of the water synthesis showed

TABLE 5

THE RECOIL ATOM EFFECT

EXPERIMENT 3. $2CO(g) + 1O_2(g) = 2CO_2(g)$
D = 1.077 CM. E_0 = 33.79 MC. RN

ACTUAL DATA

$e^{-\lambda t}$	$P_{2CO+1O_2} = \alpha$	$P'_{2CO+1O_2} = \alpha'$	$\left(\frac{k\mu}{\lambda}\right)_c$	$\left(\frac{k\mu}{\lambda}\right)'_{corr.}$	$\frac{\left(\frac{k\mu}{\lambda}\right)'_{corr.}}{\frac{\alpha' + R}{\alpha'}}$
1.0000	689.3	689.3			
0.9698	602.1	614.6	132.6	112.4	102.6
0.9111	437.4	473.4	161.2	131.6	117.9
0.8803	371.9	417.3	155.9	121.2	106.3
0.8131	255.3	317.3	165.7	120.7	103.0
0.7828	209.9	278.4	191.3	127.8	105.7
0.7520	163.1	236.3	242.5	149.5	119.5
0.7158	117.8	199.5	266.1	145.3	113.1
0.6720	76.4	164.0	292.7	132.4	98.5
0.6281	35.3	128.8	520.6	163.3	114.3
0.5828	19.4	115.2	391.2	72.9	48.3
0.5277	13.1	109.8	210.9	25.8	16.6
0.3976	9.0	106.2	86.4	7.6	4.6
0.0000	0.0	99.1	163.6	5.2	3.2
				Average	106.0

INTERPOLATED DATA

$e^{-\lambda t}$	P_{2CO+1O_2}	P'_{2CO+1O_2}	$\left(\frac{k\mu}{\lambda}\right)'_{corr.}$	$\frac{\left(\frac{k\mu}{\lambda}\right)'_{corr.}}{\frac{\alpha + R}{\alpha}}$	$\frac{\left(\frac{k\mu}{\lambda}\right)'_{corr.}}{\frac{\alpha' + R}{\alpha'}}$
1.00	689.3	689.3			
0.96	573.1	589.7	115.5	105.1	105.2
0.92	466.3	498.2	124.8	111.3	111.9
0.90	419.5	458.0	124.5	109.2	110.2
0.88	375.7	420.5	126.4	109.2	110.8
0.86	335.4	386.0	126.7	107.8	109.8
0.84	298.3	354.2	127.3	106.3	109.0
0.80	232.9	298.1	127.6	103.2	107.0
0.76	176.5	299.8	130.8	99.9	106.5
0.74	151.4	228.3	133.2	96.5	105.7
0.72	128.8	208.4	135.0	83.4	105.1
0.70	107.2	190.4	133.7	87.4	102.0
0.68	86.6	172.7	144.4	87.7	107.6
0.66	66.7	155.7	153.4	84.2	111.3
0.64	48.4	140.0	157.3	74.8	110.8

R equal to 118 mm. in a sphere of 0.9647 cm. diameter. For a sphere of 1.077 cm. diameter containing $2CO + 1O_2$ mixture, we should expect to find

$$R = \frac{118 \times 0.527 \times 0.9647}{1.03 \times 1.077} = 54.1 \text{ mm.}$$

where 0.527 and 1.03 are the specific ionizations of $2H_2 + 1O_2$ and $2CO + 1O_2$ mixtures, respectively The second calculation anticipates a result to be presented later. It is found that the average velocity constant for the first few intervals in a sphere of 1.848 cm. diameter was 37.0. If $\frac{\left(\frac{k\mu}{\lambda}\right)'_{corr.}}{\frac{\alpha' + R}{\alpha'}}$ is inversely proportional to the square of the diameter of the reaction vessel, as has been shown for the water synthesis, and if R is inversely proportional to the diameter, by using the velocity constant for the first measured interval of experiment 3, then

$$\frac{37.0 \times 1.848^{-2}}{\frac{652.0 + \frac{1.077}{1.848} R}{652.0}} = \frac{112.4 \times 1.077^{-2}}{\frac{652.0 + R}{652.0}}$$

from which we find R equals 61.9 mm. This is in excellent agreement with the preceding calculation when it is realized that as much as a ten per cent error in R will produce but a 1–2% error in the velocity constant in 1-cm. spheres, and a still smaller inaccuracy in larger vessels.

The Law of the Inverse Square of Diameters.—If chemical action in a gas or mixture of gases is proportional to the length of the mean path of the alpha particles within the volume, the velocity of reaction should increase linearly with the diameter of a sphere, other things being equal. But if the reaction is being measured by pressure change, as in the case with the present reaction, the pressure effect from a given amount of chemical action will diminish in inverse proportion to the volume or with the cube of the diameter. Combining these two effects predicts that the velocity constant measured in terms of pressure will diminish inversely with the square of the diameter.[10]

Since this relation has hitherto been tested only in the case of the synthesis of water, it appears desirable to test its validity for the oxidation of carbon monoxide. With this object in view, experiments were carried out of the same type already described except that no initial quantity of carbon dioxide was added. The results are contained in tables 6 and 7.**

Table 6 contains duplicate experiments in spheres of the same size in which it will be seen that the agreement between the mean velocity constants is satisfactory. Table 7 contains a similar experiment in a sphere of twice the diameter. It will be seen that the velocity constant now assumes a much lower value as one would expect.

TABLE 6

EXPERIMENT 4. $2CO(g) + 1O_2(g) = 2CO_2(g)$
$D = 1.844$ CM. $E_0 = 31.81$ MC. RN $R = 36.2$ MM.

$e^{-\lambda t}$	P_{2CO+1O_2}	$\left(\frac{k\mu}{\lambda}\right)'_c$	$\left(\frac{k\mu}{\lambda}\right)'_{corr.}$	$\left(\frac{k\mu}{\lambda}\right)'_{corr.} \div \frac{\alpha' + R}{\alpha'}$
1.0000	671.4			
0.9274	599.1	49.3	42.0	39.7
0.8532	534.7	48.2	40.2	37.9
0.8152	508.5	41.6	34.1	32.0
0.7605	471.4	43.6	35.4	33.1
0.6830	420.7	46.2	36.9	34.3
0.6345	392.4	45.1	35.3	33.6
0.5875	365.4	44.0	34.0	31.3
0.5347	335.5	45.7	34.5	31.7
0.4753	311.4	47.4	35.1	32.0
0.4355	295.3	41.9	31.7	28.7
0.3348	252.3	49.1	34.9	31.4
0.2756	230.8	47.3	32.3	28.8
0.2321	214.0	54.6	36.3	32.2
0.1807	191.1	69.2	44.5	39.2
0.0000	114.4	89.3	50.8	43.7
			Average	35.6

EXPERIMENT 5. $2CO(g) + 1O_2(g) = 2CO_2(g)$
$D = 1.848$ CM. $E_0 = 41.97$ MC. RN $R = 36.1$ MM.

$e^{-\lambda t}$	P_{2CO+1O_2}	$\left(\frac{k\mu}{\lambda}\right)'_c$	$\left(\frac{k\mu}{\lambda}\right)'_{corr.}$	$\left(\frac{k\mu}{\lambda}\right)'_{corr.} \div \frac{\alpha' + R'}{\alpha'}$
1.0000	848.5			
0.9418	731.6	60.7	51.5	49.2
0.6770	629.5	55.3	45.7	43.5
0.8395	563.4	48.3	36.5	34.5
0.7985	539.9	45.1	36.0	34.0
0.7606	499.0	49.5	38.9	36.5
0.7189	458.5	48.4	37.4	35.0
0.6762	421.3	47.2	35.6	33.2
0.6368	388.1	49.7	36.8	34.2
0.5945	357.8	45.8	33.2	30.7
0.5447	320.2	53.1	36.4	33.5
0.4941	283.4	57.5	39.1	35.7
0.4508	256.9	54.0	35.4	32.1
0.3931	221.9	60.5	38.0	34.2
0.2777	163.2	64.7	37.1	32.9
0.1982	127.4	72.4	36.4	31.7
0.1132	96.5	77.9	35.4	30.4
0.0658	82.1	81.3	31.5	26.6
0.0000	63.8	91.3	30.8	25.7
			Average	36.9

Table 8 contains a summary giving the average velocity constant in spheres differing by four-fold in diameter. The final column contains a constant corrected according to the Inverse Square of the Diameter Law which shows satisfactory agreement. The difference between the third and fourth

TABLE 7

EXPERIMENT 6. $2CO(g) + 1O_2(g) = 2CO_2(g)$
$D = 3.940$ CM. $E_0 = 67.0$ CM. RN $R = 16.1$ MM.

e^{-M}	P_{2CO+1O_2}	$\left(\frac{k\mu}{\lambda}\right)'_c$	$\left(\frac{k\mu}{\lambda}\right)'_{corr.}$	$\left(\frac{k\mu}{\lambda}\right)'_{corr.} \div \frac{\alpha'+R}{\alpha'}$
1.0000	666.6			
0.9139	609.9	15.27	13.01	12.68
0.8673	582.7	14.45	12.17	11.84
0.8167	555.1	14.22	11.91	11.57
0.7577	527.2	12.92	10.72	10.40
0.6856	497.1	12.06	9.91	9.60
0.6188	469.3	12.74	10.35	10.02
0.5698	450.3	12.48	10.05	9.71
0.4868	421.9	11.61	9.23	8.91
0.4158	399.3	11.47	9.04	8.71
0.3474	378.1	11.79	9.19	8.84
0.1729	322.0	13.61	10.23	9.81
0.1035	301.0	14.37	10.59	10.11
0.0672	290.5	14.47	10.39	9.90
0.0202	277.5	14.38	10.39	9.90
0.0000	271.6	15.73	10.84	10.30
			Average	9.65

columns involves correction for differing average intensities of ionization (F) in different sized spheres. This correction is discussed by Glockler and Heisig[11] and is significant in large spheres.

TABLE 8

TEST OF THE INVERSE SQUARE LAW

D	$\left(\frac{k\mu}{\lambda}\right)'_{corr.} \div \frac{\alpha'+R}{\alpha'}$	$\frac{\left(\frac{k\mu}{\lambda}\right)'_{corr.} \times D^2}{\frac{\alpha'+R}{\alpha'}}$	$\frac{\left(\frac{k\mu}{\lambda}\right)'_{corr.} \times D^2}{\frac{\alpha'+R}{\alpha'} \times F}$
1.077 cm.	106.0	123.0	123.0
1.844	35.6	121.1	120.0
1.848	36.9	126.0	124.1
3.936	9.65	149.0	136.8

Applying all of the corrections developed in this paper to the results of Lind and Bardwell,[12] i.e., the catalytic effect of carbon dioxide and the recoil atom effect, gives values for the velocity which have an average value the same as that of the present work for the same sized sphere, but which individually show a marked downward trend for which no explanation has been found.

Calculation of the M/N Ratio.—The method of calculating M/N ratio has been fully presented in several publications. We may state the result applied to the case, where M molecules of CO and O_2 are disappearing per N ion pairs produced in these gases by alpha particles, as:

$$-\frac{M_{(2CO+1O_2)}}{N_{(CO+O_2)}} = \frac{\left(\frac{k\mu}{\lambda}\right)'_{corr.} 2\lambda VA}{i \cdot F \cdot H \cdot 0.61D \cdot 11.16 \times 10^{10}}$$

where V and D are the volume and diameter of the spherical reaction vessel, A is Loschmidt's number, i is the specific molecular ionization of the stoichiometric reaction mixture, H is the correction factor for recoil atoms $\frac{\alpha' + R}{\alpha'}$, λ is the decay constant of radon, F is the average intensity of ionization and $\left(\frac{k\mu}{\lambda}\right)'_{corr.}$ is the velocity constant.

The quantity 11.16×10^{10} represents the total number of alpha particles emitted by radon in equilibrium through radium C, i.e., $3 \times 3.72 \times 10^{10}$, where 3.72×10^{10} is the number of alpha particles emitted each second per gram of radium.[13] The quantity $0.61\frac{D}{2}$ ($= 0.61\,R$ where R is the radius) is the average path of an alpha particle.[14] The values of i for carbon monoxide and oxygen are 100 and 110, respectively; hence, for the stoichiometric mixture $i = 1.03$. The form of the recoil correction has already been discussed. The value of A is 2.705×10^{19} cm.$^{-3}$. For λ we shall use 2.099×10^{-6} sec.$^{-1}$.[15] A series of F values for different pressures is taken from the recent calculations of the average number of ion pairs produced in air per centimeter of path per alpha particle, made by Glockler and Heisig[16] using Mund's[17] theoretical calculation of ionization produced by radon.

Grouping all terms constant throughout any given experiment, we have

$$-\frac{M_{(2CO+1O_2)}}{N_{(CO+O_2)}} = \frac{\left(\frac{k\mu}{\lambda}\right)'_{corr.}}{F \cdot H} \cdot \frac{2\lambda\, A\, V}{11.16 \times 10^{10} \cdot 0.61D \cdot i}$$

$$= \frac{\left(\frac{k\mu}{\lambda}\right)'_{corr.}}{F \cdot H} \cdot \frac{V}{D \cdot i} \times 1.668 \times 10^{3}.$$

Table 9 contains a sample calculation of the M/N ratio for the carbon monoxide oxidation.

For experiments 3 and 5 the average values of the $-M/N$ ratio are 3.76 and 3.83. It is clear that the value of this ratio is between three and four in the carbon monoxide oxidation. It is of interest to note that a study of this reaction in a glow discharge produces a similar yield.[18]

Mechanism of the Carbon Dioxide Effect.—It has been quite difficult to find a wholly acceptable mechanism for any of the gas ion reactions. This difficulty is only enhanced in the case of the addition of an inert gas and its ions.

TABLE 9

CALCULATION OF M/N RATIO FOR EXPERIMENT 4

$1.03P_{2CO+1O_2}$	F	H	$\left(\frac{k\mu}{\lambda}\right)'_{corr.}$	$\frac{M_{(2CO+1O_2)}}{N_{(CO+O_2)}}$
691.5	2.83×10^4	1.054		
617.1	2.82	1.059	42.0	4.05
550.1	2.81	1.065	40.2	3.89
523.8	2.80	1.068	34.1	3.29
485.5	2.80	1.072	35.4	3.41
433.3	2.79	1.079	36.9	3.55
404.2	2.78	1.084	35.3	3.38
376.4	2.78	1.088	34.0	3.25
345.6	2.77	1.094	34.6	3.29
320.7	2.77	1.100	35.1	3.38
304.2	2.76	1.104	31.7	3.00
259.9	2.76	1.116	34.9	3.28
237.7	2.76	1.123	32.3	3.02
220.4	2.75	1.130	36.3	3.38
196.8	2.75	1.139	44.5	4.11
117.8	2.74	1.187	50.8	4.59
			Average	3.69

Two general types of mechanism have been proposed: (1) *Ion-clustering* in which it is assumed that the reactant molecules are brought together under the electrodynamic attraction of a central charge (positive and sometimes also negative ions) and that chemical reaction takes place upon neutralization of the charge by recombination of oppositely charged ions.

(2) *Dissociation of Ions.*—Either upon formation or neutralization of the ions, free atoms or radicals are formed which are chemically reactive.

The mechanism of the effect of an inert gas ion is fairly obvious for the first class if one assumes clustering about the foreign ion itself, which seems to present no electrodynamic difficulties. But in order for the foreign ion to exert an influence in the second type of mechanism, it is necessary to assume transfer of its energy by collision of the second kind, either during its life as an ion or immediately following its electrical neutralization, when the heat of ion recombination may be available.

The experimental evidence both in this reaction and in other reactions when CO_2 is present or produced indicates that CO_2 may have two oppositely directed effects. Its ions may tend to accelerate the ionic reaction, while its neutral molecules may have a retarding effect, which might be interpreted in the clustering mechanism as a "crowding out" or inefficiency effect due to CO_2 in the ion clusters. In the second hypothesis, the retarding effect of CO_2 molecules would be due to a "quenching" effect, that is, to an inefficient utilization of energy received by exchange. Too little is known of these effects to attempt at present to put them into the form of more definite mechanisms.

Summary.—The purpose of this paper was to ascertain the extent to which carbon dioxide influences the rate of the alpha-ray reaction in which it is produced by the union of oxygen and carbon dioxide. Only about 14.5% of the ionization falling upon CO_2 is found to be chemically effective.

The recoil atom effect on the oxidation of carbon monoxide was examined in a one-centimeter sphere. The correction to be applied to velocity constants was $\frac{1}{\frac{\alpha' + R}{\alpha'}}$ where α' is the sum of the partial pressures of reactants and the ionic equivalent pressure of the catalyst. The value of R found to be 62 mm. is in agreement with that predicted from the water synthesis.

The inverse square of the diameter law was shown to hold for this oxidation by comparing the values of $\left(\frac{k\mu}{\lambda}\right)'_{corr.} \div \frac{\alpha' + R}{\alpha'} \times D^2$ in spheres of several sizes. An average value of about *126* was obtained.

The M/N ratio for the reaction was calculated to be between three and four.

* Part of a Ph.D. Thesis submitted by Charles Rosenblum, University of Minnesota (1931).

** $\left(\frac{k\mu}{\lambda}\right)'_c$ is the velocity constant calculated on the basis of partial pressure of reactant, without correction for the ionic equivalent of carbon dioxide; $\left(\frac{k\mu}{\lambda}\right)'_{corr.}$ is the constant corrected for the fractional catalytic effect of the dioxide; $\left(\frac{k\mu}{\lambda}\right)'_{corr.} \div \frac{\alpha' + R}{\alpha'}$ is the latter constant corrected as described in the preceding section for the recoil atom effect.

[1] S. C. Lind and D. C. Bardwell, *J. A. C. S.*, **48**, 1575 (1926).

[2] S. C. Lind and D. C. Bardwell, *Ibid.*, **47**, 2686 (1925).

[3] See footnote 1, p. 1582.

[4] S. C. Lind and D. C. Bardwell, *J. A. C. S.*, **46**, 2005 (1924).

[5] S. C. Lind, *J. A. C. S.*, **41**, 531 (1919).

[6] S. C. Lind and D. C. Bardwell, *J. A. C. S.*, **48**, 1575 (1926).

[7] In most alpha-ray reactions, the velocity is somewhat irregular at the start. Accordingly the value of x is not calculated for the first interval.

[8] S. C. Lind, *J. A. C. S.*, **41**, 551 (1919).

[9] *Loc. cit.*

[10] S. C. Lind and D. C. Bardwell, *J. A. C. S.*, **46**, 2005 (1924).

[11] G. Glockler and G. B. Heisig, *J. Phys. Chem.*, **35**, 2478 (1931).

[12] S. C. Lind and D. C. Bardwell, *J. A. C. S.*, **47**, 2686 (1925).

[13] Hess and Lawson, *Wien. Ber.*, **127**, 405 (1918).

[14] S. C. Lind and D. C. Bardwell, *J. A. C. S.*, **45**, 2585 (1923).

[15] Curie and Chamie, *J. de Physique et Radium*, **5** [6], 238 (1924).

[16] See footnote 11.

[17] Mund, *J. Phys. Chem.*, **30**, 890 (1926).

[18] Brewer and Kueck, *Ibid.*, **35**, 1281 (1931).

"FEATHER JOINTS" AS INDICATORS OF THE DIRECTION OF MOVEMENTS ON FAULTS, THRUSTS, JOINTS AND MAGMATIC CONTACTS

BY ERNST CLOOS

GEOLOGICAL DEPARTMENT, THE JOHNS HOPKINS UNIVERSITY

Communicated March 30, 1932

A certain type of local joints, which appear frequently along boundaries of moved blocks or contact zones of batholitic intrusions, have been described in abundance in recent years. [1,2,3] The name "feather joints" was given as a merely descriptive term because these joints are arranged in a way similar to the barbs of a feather in relation to its shaft.

Van Hise[4] shows some examples of "feather joints," formed under rotational deformation, in the pre-Cambrian quartzites in Wisconsin. He discusses the phenomena as fissility in relation to cleavage. As these joints are restricted to certain zones along which movements have taken place and are closely related to such movements, they have gained more and more importance during recent years, especially in connection with structural observations in granites.

It is possible to give clear evidence of movements in a definite direction with the help of such joints. Several examples are described below. A series of experiments was conducted in order to throw some light on the origin and interpretation of "feather joints" especially in relation to the stress-strain ellipsoids.

The relationship of fracture cleavage, as described by A. Leith[5] and C. K. Leith,[6] and "feather joints" is of great interest, since fracture cleavage serves a similar purpose. According to Leith it is possible to use fracture cleavage to interpret the direction of relative movements between certain formations in folded stratified rocks. The author believes that there is a contradiction in the interpretations of feather joints and fracture cleavage, especially in the application of the strain ellipsoid. The discussion of this contradiction will follow later; the present paper is restricted to the application of "feather joints."

Experiments.—The experiments described below were carried out in the Geological Laboratory of the Johns Hopkins University in Baltimore, in the same manner as the experiments of Hans Cloos[7–13] in Bonn. A paper about some of the writer's experiments, illustrated by motion pictures, was presented at the 44th annual meeting of the Geological Society of America, at Tulsa, Oklahoma.

Wet clay affords a very favorable material for tectonic experiments. The strength of wet clay is in direct proportion to the strength of certain parts of the earth's crust and to the reduction in scale which is necessary in

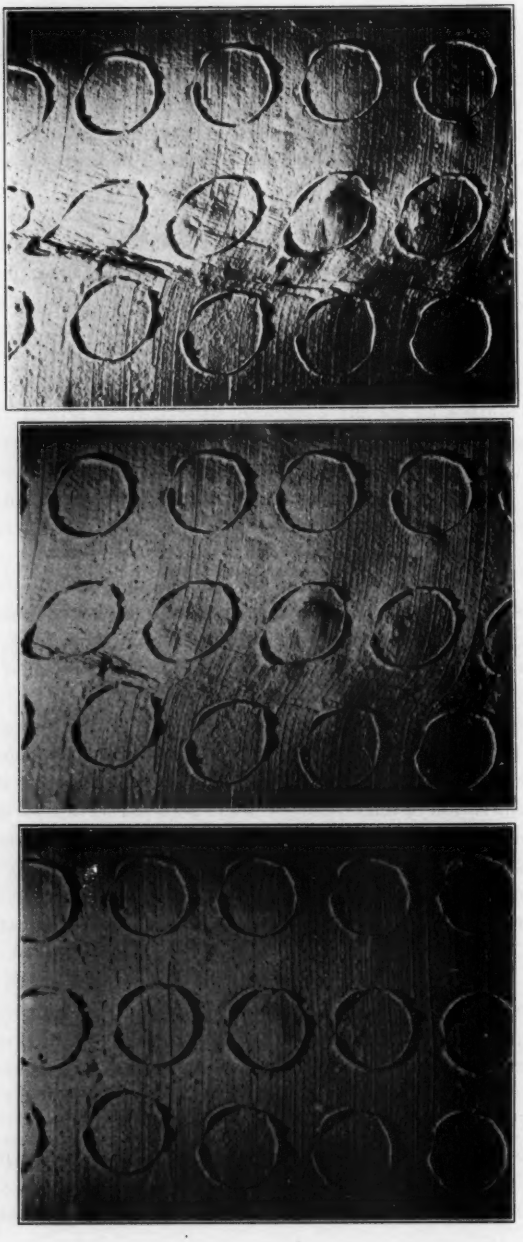

FIGURE 1 FIGURE 2 FIGURE 3

Three photographs of a clay experiment in three succeeding stages.

order to reproduce the tectonic features of a structural unit as observed in the field. A good light shining at a very low angle to the clay surface, will show thousands of joints in deep relief.

The first experiment was carried out as follows:

Two sheets of iron, one movable, the other stationary, were placed upon a table side by side and covered with a homogeneous clay cake, 20 inches long, 10 inches wide and 5 inches thick. Circular impressions were made upon the surface of the clay cake to indicate the direction of deformation and the axis of strain. (See Figs. 1–3.) The writer is aware of the fact that these circles indicate two directions of the deformation only, but he believes that they are useful for illustration, if the assumption is made that a circle on the surface represents a projection of a sphere upon that plane. The arrangement described above makes it possible to photograph the orientation of the ellipsoid and to make measurements without any deductive or hypothetical constructions.

The axis of the ellipsoid, shear planes and joints can be copied from a photograph of the deformed clay cake, studied in their mutual relations, and compared with conditions found in nature. For the latter purpose it is convenient to draw the results obtained from experiments upon transparent paper, which can be superimposed upon photographs of structural elements as observed in the field. If the directions coincide the confirmation becomes complete.

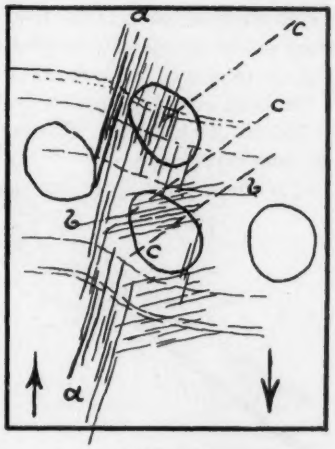

FIGURE 4

The main features of figure 2 redrawn for better understanding and easier discussion. Dotted line (*c*) represents direction of tension joints at an angle of approx. 45° to the direction of movement. Joint systems (*bb*) and (*aa*) are shear joints at an angle of approximately 30° with (*c*). The direction of movement trends parallel to the frame lines of the figure (see arrows).

The lines seen upon the surface of the clay which are produced in smoothing the cake indicate the direction and amount of movement and deformation.

Figure 4 was drawn from the original photograph (Fig. 2) for greater ease of discussion and comprehension.

If the movable iron sheet upon which part of the clay rests, is moved in the direction represented by the arrows in figure 4, a certain rotational deformation takes place above the contact zone between the stationary and

the moved part of the cake. The width of that zone depends upon the thickness of the clay cake. (Riedel,[14,15] Rüger.[16])

The longest axis of the ellipsoid forms a certain angle with the direction of movement and the following systems of joints appear:

1. *Tension Joints* (*System cc Fig. 4*).—Tension joints form an angle of 90 degrees with the longest axis of the ellipsoid, an angle of approximately 45 degrees with the direction of movement, or "pull" on the iron sheet, and an angle of 45 degrees with the edge of the moved sheet. All measurements must be taken at the first indication of jointing, because the directions will be turned into complex and varied positions afterward. (See 14, Fig. 5, rotational deformation.)

The long axis of the resulting ellipsoid coincides with the axis of folding in Mead's illustration.[17] Folding of the clay was observed occasionally in the same experiment.

It is advisable to spray the surface of the clay cake with a thin film of water as tension joints will then open more easily. Tension joints are always the first joints to open in experiments of this kind. (See Mead,[17] Fath,[18] Riedel.[15])

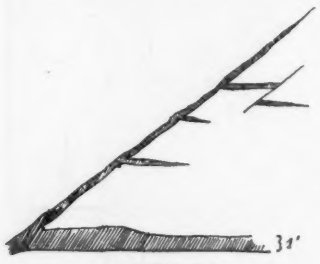

FIGURE 5

Lamprophyric dike with smaller dikes, branching off the main dike in the direction of "feather joints." Yosemite Natl. Park.

2. *Shear Joints* (*Systems bb and aa in Fig. 4*).—The system *aa* in figure 4 forms an angle of 30 degrees (not 45!) with the tension joints and an angle of approximately 60 degrees with the longest axis of the ellipsoid. The system *bb* forms approximately the same angle on the other side of the tension joints, thus being symmetrical with the system *aa* in reference to (*c*). Therefore the angle of 60 degrees between the two systems *aa* and *bb* points toward the direction of shortening, which is inferred as normal to the longest axis of the ellipsoid. (See Bucher,[19] Figs. 3, 6, 8.)

If the experiment is carried further, the system *aa* serves as a zone of horizontal displacement. The single faults are "en echelon" faults and *not* parallel to the actual movement. (See Hans Cloos,[7-13] Riedel.[15]) The ellipsoid is not deformed further but is broken up into a number of pieces which continue their movements along joint and fault system *aa*. (Fig. 3.)

It is very noticable that the faults do not trend parallel to the actual movement, e.g., parallel to the pull and the edges of the iron sheet. Similar fault zones are found in abundance in nature.

If the experiment is carried on still further, large shear zones develop in the direction *aa* until the clay cake is finally torn apart. However, the

first joints to open and the first faults to form are parallel to that later direction. Thus, in the writer's experiments the direction of maximum shear was found to form an angle of approximately 15 degrees to the edges of a square which is deformed into a rhomboid. Examples of natural conditions will be described below.

3. *Folds.*—Several of the writer's experiments show folding by rotational strain in the same orientation as described by Leith, Mead and others, that is, parallel to the longest axis of the ellipsoid.

Natural Examples.—"Feather joints," for the most part, seem to follow the direction of tension joints, direction *c* in our experiments. Examples of feather joints have been found at many localities, e.g., in the Yosemite region (California).

Figure 5 shows a lamprophyric dike in the Sierra Nevada granodiodiorite. The main dike follows the general direction of jointing, being approximately N–S. Small offshoots of the dike have filled other joints which have opened at an angle of 30 degrees to the main dike. The apex of the angle points in the direction of movement.

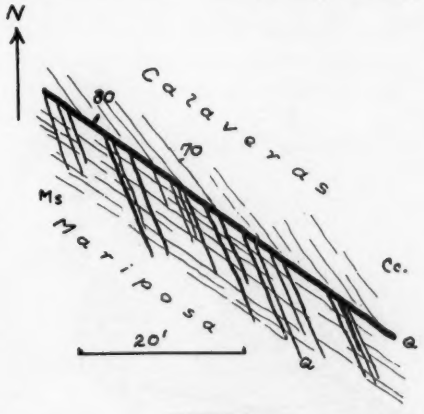

FIGURE 6

Contact between Mariposa (Jurassic) and Calaveras (Carboniferous) formation. *Q* = quartz dikes. Smaller joints which branch off the main contact-dike, show the position of "feather joints" (approx. 30°). The movement east of the contact is directed relatively toward the southeast. Mormon Creek near Tuttletown, California.

Figure 6 represents a map of the contact between the Calaveras and the Mariposa formations as exposed in the bed of Mormon Creek near Tuttletown, California. The contact itself is represented by a quartz dike of approximately 10 inches in thickness. "Feather joints" were formed at an angle of about 30 degrees to the contact, the acute angle pointing toward the north and indicating a relative movement of the Mariposa formation toward the northwest.

Similar observations have been made many hundred times and actual displacements in that direction have also been observed in the Yosemite region. "Feather joints" are always linked with such displacements and pointed in the direction of the movement. Gradually, the idea was impressed upon the writer that the western part of the Sierra Nevada shows movements toward the north or the northwest, in relation to the eastern

part, thus in the same direction as recent movements, which are connected with the great faults of California. Additional studies are expected to show the magnitude and importance of these movements.

Marginal thrusts, representing movements toward the contacts of igneous masses were observed in the Sierra Nevada along the eastern contact of the Batholith and other localities, always in connection with a contact zone. Figure 7 shows one of the marginal thrusts which was observed on the western contact of the eastern main mass of the Sierra Nevada Batholith (see Ernst Cloos[20]), as exposed on the Ledge Trail from Yosemite Valley to Glacier Point. The thrust plane dips at an angle of 25 degrees east. The movements took place toward the west. This movement

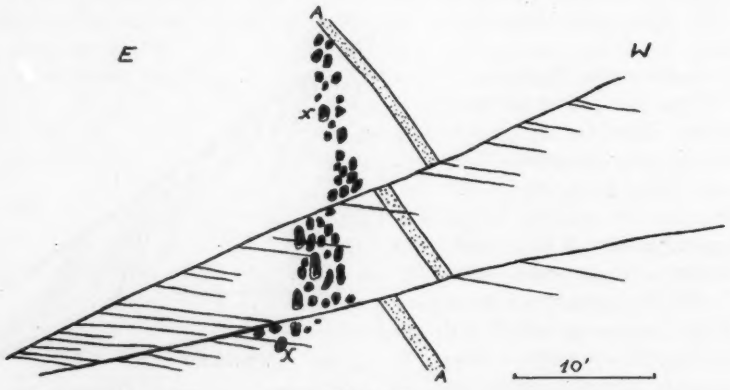

FIGURE 7

Marginal thrusts toward the western contact of the eastern "main mass" of the Sierra Nevada Batholith. Ledge Trail from Camp Curry to Glacier Point, Yosemite Natl. Park.

A = Aplite, *X* = Xenoliths. The thrust movement has caused the development of "feather joints" in the underlying part of the granodiorite.

produced feather joints in the underlying part, again at an angle of 30 degrees to the thrust plane and dipping approximately 5–10 degrees west. The thrust plane itself follows the system *bb* in the experiment with reference to the movement of the magma along the contact.

The Mother Lode.—A study of the conditions in the Mother Lode area between Mariposa and Sonora, California, and at several localities further north, as well as in the literature (see Knopf,[21] Ransome, Lindgren), has convinced the writer that this zone represents a zone of movements, in which the principle of feather joints and marginal thrusts can be applied in a very satisfactory manner.

The Mother Lode itself represents a zone of quartz veins which contain

gold in commercial quantities. Steeply and flat eastward dipping veins, with thrust movements toward the west, are described in the literature cited above (see also H. Cloos,[1] Fig. 5, p. 258).

The zone, as observed in many exposures, e.g., at Quartz Mountain, near Sonora or in road cuts west of Jamestown, is represented by one or several large quartz dikes, which dip steeply to the east. Quartz-filled joints, striking approximately parallel to the dikes and dipping about 35–50 degrees east, are observed, as well as movements toward the west on these joints. More gently dipping joints and dikes are rare but were observed occasionally. It appears to the writer that the first system resembles the direction *aa* in the experiment. The second system would then follow the orientation of tension joints. As natural conditions are not as regular as experiments, the directions may vary slightly from place to place, but the general relationship seems to be the observed one.

Figure 8 gives a picture of these relations on the north side of Quartz

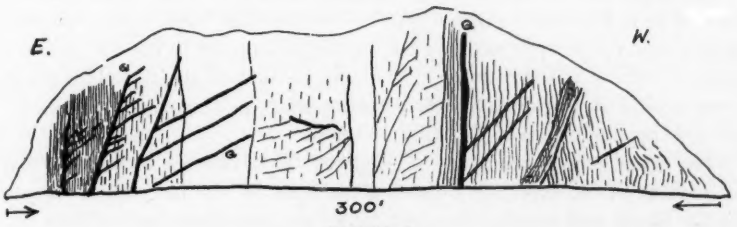

FIGURE 8

Section across the Mother Lode, Quartz Mountain, near Jamestown, California. Smaller joints branch off large steep dipping quartz dikes. The wall rock is composed mainly of altered Mariposa slate and Calaveras schist.

Mountain, southwest of Sonora. Large quartz dikes dip steeply east; and smaller joints, opened at an angle of 30 degrees, to the large dikes are also filled with quartz. A large number of such joints was observed in the granodiorite of Groveland and Big Oak Flat, where several hundred small quartz dikes follow the direction *c* of the experiment.

The details of the study made by the writer cannot be discussed here, but will be published later. The results may be summarized as follows:

The Mother Lode zone represents an upthrust zone. A rotational stress was exerted during the upthrust movement and certain systems of joints were opened. These are similar to the systems *aa*, *bb* and *c* in the experiments referred to above. The joints, especially system *c*, were filled with gold quartz veins.

The intrustion of large masses of granodiorite east of the Mother Lode is thought by the writer to be responsible for the thrust-like movements and

the vein filling. Furthermore, the magna intrusion caused the thrust movements along a large number of gently dipping quartz veins, in a manner similar to that by which marginal thrusts were formed, which were observed especially in the vicinity of the eastern contact of the Sierra Nevada Batholith. The writer is inclined to believe therefore, that the Mother Lode zone represents the western *dynamic* contact of the Sierra Nevade intrusion. The bearing of these observations upon economic problems in the area will be discussed in a later paper.

Conclusions.—Examples of "feather joints" could be added in unlimited number. No contradictory facts have been found as far as the writer is aware. It is, therefore, believed that the appearance of such joints can be taken as an established fact and as such be used as a valuable tool for the determination of relative movements between large or small displaced blocks. A "tendency" toward movement seems to be sufficient to produce "feather joints," even in cases where the actual displacement is very small.

The angle of approximately 30 degrees, which was observed in the field as well as in the experiments, supports the idea that such joints represent local tension joints produced by the rotational impulse in the vicinity of a zone of differential movements. The orientation of the complete set of tension and shear joints in the experiment (*cc*, *bb*, *aa*) gives valuable aid for the application of the strain ellipsoid.

The shear joint systems at an angle of 12–15 degrees with the actual movement in the experiment throw some light on the fact that so many of the larger and smaller observed faults are not single fault planes but systems of faults. The members of these systems are generally arranged "en echelon," at a small angle to the trend of the fault zone and succeed one another.

The close coincidence between the results of the author's and Hans Cloos' experiments and the conditions found in nature, expressed in similar angles between joints or faults, lend proof to the assumption, made at the beginning of this paper, that wet clay, mixed with water to the proper consistency, affords a material which permits the imitation of natural conditions in the laboratory. Such imitations will certainly always suffer from the impossibility of exactly reproducing nature with respect to time, space and acting forces. But whenever such imitations are attempted the ratios of these factors ought to be maintained as closely as possible.

[1] Cloos, Hans, "Bau und Bewegung der Gebirge in Nordamerika, Skandinavien und Mitteleuropa," *Fortschr. d. Geol. a. Pal.*, Bd., VII, H. 21; p. 254, fig. 3; 258, fig. 5; 260, fig. 6 (1928).

[2] Cloos, Ernst, "Tektonik des Granits von Gorkau Kr. Nimptsch in Schlesien," *Abh. d. Pr. Geol. L. A. H.*, **89**, p. 96, figs. 24–27 (1922).

[3] Ljungner, E., "Spaltentektonik und Morphologie der schwedischen Skagerrakküste," *Bull. Geol. Inst. Univ. Upsala*, XXI **21** (1930).

[4] van Hise, C. H., "Principles of North American Pre-Cambrian Geology," *16th Ann. Rep. U. S. G. S.*, p. 581 (1894).

[5] Leith, Andrew, "The Application of Mechanical Structural Principles in the Western Alps," *J. Geol.*, **39**, No. 7, 631 (1931).

[6] Leith, C. K., *Structural Geology*, Henry Holt, New York, p. 154, figs. 57–60 (1923).

[7] Cloos, Hans, "Künstliche Gebirge," *Natur u. Museum*, H. 5 (1929), Senkenbergische Naturforschende Gesellschaft, Frankfurt a. Main.

[8] Cloos, Hans, "Künstliche Gebirge. II," *Natur u. Museum*, H. 6 (1929).

[9] Cloos, Hans, "Zur experimentellen Tektonik," *Naturwissenschaften*, **18**, H. 34 (1930).

[10] Cloos, Hans, "Zur experimentellen Tektonik," *Geol. Rundschau*, **21**, H. 6 (1930).

[11] Cloos, Hans, "Zur experimentellen Tektonik," *Naturwissenschaften*, **19**, H. 11 (1931).

[12] Cloos, Hans, "Einige Versuche zur Granittektonik," *Neues Jahrb. Mineral. Geol.*, **64**, Abt. B, 829–836 (1931).

[13] Cloos, Hans, "Fliessen und Brechen in der Erdkruste und im geologischen Experiment," *Plastische Massen*, 1 (1931).

[14] Riedel, W., "Das Aufquellen geologischer Schmelzmassen als plastischer Formänderungsvorgang," *N. Jahrb. Beil. Bd.*, **67**, Abt. B p. 151 (1929).

[15] Riedel, W., "Zur Mechanik geologischer Brucherscheinungen ein Beitrag zum Problem der 'Fiederspalten,' " *Centralblatt Min. Geol. Pal.*, Abt. B (1929).

[16] Rüger, L., "Die Untersuchungsergebnisse an Gesteinsdeformationen (Petrotektonik)," *Geol. Rundschau*, **22**, H. 2 (1931).

[17] Mead, W. J., "Notes on the Mechanics of Geologic Structures," *J. Geol.*, **28**, p. 505 (1920).

[18] Fath, A. E., "The Origin of the Faults, Anticlines and Buried Ridges of the Northern Part of the Mid-Continent Oil and Gas Field," *U. S. G. S. Prof. Paper*, **128**, 75–85 (1921).

[19] Bucher, W. H., "The Mechanical Interpretation of Joints," *J. Geol.*, **28**, 707 (1920).

[20] Cloos, Ernst, "Der Sierra Nevada Pluton," *Geol. Rundschau*, **22**, H. 6, 380 (1931).

[21] Knopf, A., "The Mother Lode Zone of California," *U. S. G. S., Prof. Paper*, **157**, 76 (1929).

Additional literature, not mentioned in the text:

Sander, B., *Gefügekunde der Gesteine*, Julius Springer, Vienna (1930).

Melton, F. A., "A Reconnaissance of the Joint Systems in the Quachita Mountains," *J. Geol.*, **37** (1929).

Fujiwhara, S., "On the Echelon Structure of Japanese Volcanic Ranges and Its Significance from the Vortical Point of View," *Gerlds. Beitr. z. Geophysik.*, **16** (1927).

Willis, Bailey, "Geologic Structures," McGraw-Hill, New York, p. 143, fig. 73 (1929).

Morse, R. R., "A Note on Jointage and the Application of the Strain Ellipsoid," *J. Geol.*, **31**, 669 (1923).

Cloos, Ernst, "Structural Survey of the Granodiorite South of Mariposa, California," *Am. J. Sc.*, (1932).

ON CERTAIN CRITERIA FOR FOURIER CONSTANTS OF L INTEGRABLE FUNCTIONS

BY CHARLES N. MOORE

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CINCINNATI

Communicated April 14, 1932

The relationship between functions whose squares are L integrable and their corresponding Fourier constants is extremely simple and elegant. A combination of the theorems of Parseval and Riesz-Fischer tells us that the convergence of the series $\Sigma(a_n^2 + b_n^2)$ is both a necessary and sufficient condition that there should be a function of integrable square from which these constants may be obtained. For the more general case of L integrable functions we have no such simple result. A necessary condition is $\lim_{n \to \infty} a_n = 0 = \lim_{n \to \infty} b_n$, but this is not sufficient.

Recently Salem has considered the case where the formal integral of the trigonometric series whose coefficients are a given set of constants converges to a continuous function, and he has established necessary and sufficient conditions that the set of constants should be the Fourier constants of an L integrable function.[1] While these conditions are undoubtedly of theoretical importance, it would appear to be difficult to apply them directly as criteria to a particular set of constants. The purpose of the present note is to obtain certain sufficient conditions that a set of constants (a_n) should not be in the first instance, and should be in the second instance, the Fourier constants of an even L integrable function. It will be shown by simple examples that in a number of cases these criteria can be used directly to test the nature of a given set of constants.

We shall obtain first the criterion that a set of constants should not be the Fourier constants of an L integrable function. Let $f(x)$ be an even function of bounded variation in $(-\pi, \pi)$ such that at least one out of every k successive Fourier coefficients is positive and we have for these positive coefficients the inequality

$$\frac{\rho}{n} \leqq a_n \leqq \frac{R}{n} \begin{pmatrix} \text{all } n \text{ where } a_n > 0 \\ R > \rho > 0 \end{pmatrix}. \tag{1}$$

Represent the subscripts of the successive positive coefficients by m_1, $m_2, \ldots, m_n, \ldots$. Then we have the inequality

$$m_n \leqq nk. \tag{2}$$

Let $\varphi(q)$ be a positive function of q, defined for $q \geqq m_r$, where r is some positive integer, that is monotonic increasing and becomes infinite with q, and is such that

$$\sum_{q=1}^{\infty} \frac{1}{q\varphi(qk)} \tag{3}$$

is a divergent series. Then the trigonometric series

$$\sum_{n=1}^{\infty} \frac{1}{\varphi(m_n)} \cos m_n x \tag{4}$$

is not the Fourier series of an L integrable function.

For suppose it were such a series, and represent by $F(x)$ the corresponding L integrable function. Form

$$g_s(x) = {}^1/_2 a_0 + a_1 \cos x + a_2 \cos 2x + \ldots + a_{ms} \cos m_s x, \tag{5}$$

where the a's are the Fourier coefficients of $f(x)$. Then, $F(x)$ being L integrable and $g_s(x)$ of bounded variation, we have

$$\int_{-\pi}^{\pi} F(x) g_s(x) dx = \sum_{n=1}^{n=s} \frac{a_{m_n}}{\varphi(m_n)}. \tag{6}$$

Since $f(x)$ is of bounded variation, $g_s(x)$ remains bounded for all s and the left-hand side of (6) is likewise bounded for all s. But the right-hand side, in view of inequalities (1) and (2) and the divergence of (3), becomes infinite with s. Hence the coefficients of (4) are not the Fourier constants of an L integrable function.

If the m_n are equally spaced, the series (4) can be shown to converge at all except a finite number of points of the interval $(-\pi, \pi)$. In the neighborhood of one or more of these points the function to which the series converges becomes infinite in such a manner as to cause its integral to diverge. When $\varphi(n)$ is a sufficiently simple function of n, we can actually sum the series by means of a contour integral and thus obtain in explicit form the function defined by the series. This procedure is feasible in the case of the following series:

$$\frac{\cos 3x}{\log 3} + \frac{\cos 7x}{\log 7} + \frac{\cos 11x}{\log 11} + \ldots\ldots\ldots$$

$$\frac{\cos 7x}{\log 7} + \frac{\cos 13x}{\log 13} + \frac{\cos 20x}{\log 20} + \ldots\ldots\ldots$$

These examples are of interest because the series $\sum_{n=2}^{\infty} (\cos nx)/\log n$ is known to represent a function which is L integrable over any finite interval. They are obtained from our theorem by choosing as $f(x)$ the functions whose cosine developments are, respectively,[2]

$$\tfrac{1}{4}\pi - \cos x + \tfrac{1}{3}\cos 3x - \tfrac{1}{5}\cos 5x + \ldots,$$

$$\cos x - \tfrac{1}{5}\cos 5x + \tfrac{1}{7}\cos 7x - \tfrac{1}{11}\cos 11x + \ldots.$$

We turn next to the criterion that a set of constants should be the Fourier constants of a function that is L integrable. It may be stated as follows:

THEOREM: *If the set of constants* $\tfrac{1}{2}a_0, a_1, a_2, \ldots,$ *satisfy the conditions*

$$\text{(A)} \qquad |a_n \log n| < M \qquad (n \geqq 2),$$

$$\text{(B)} \qquad \sum_{n=m}^{n=p} n_n |\Delta^2 a_n| < K \qquad (p > m \geqq 1),$$

where M *and* K *are positive constants then they are the Fourier constants of an* L *integrable function.*

If we set

$$s_n(x) = \tfrac{1}{2} + \cos x + \ldots + \cos nx,$$

$$S_n(x) = s_0(x) + s_1(x) + \ldots + s_n(x),$$

$$\sigma_n(x) = \tfrac{1}{2}a_0 + a_1 \cos x + \ldots + a_n \cos nx,$$

we have the identity

$$\sigma_m(x) = \sum_{n=0}^{n=m-1} S_n(x)\,\Delta^2 a_n + S_m(x)\,\Delta a_m + s_m(x) a_{m+1}$$

$$= \tfrac{1}{2}\left[\sum_{n=0}^{n=m-1} \frac{\sin^2 \tfrac{1}{2}(n+1)x}{\sin^2 \tfrac{1}{2}x}\,\Delta^2 a_n + \frac{\sin^2 \tfrac{1}{2}(m+1)x}{\sin^2 \tfrac{1}{2}x}\,\Delta a_m + \right.$$

$$\left. \frac{\sin \tfrac{1}{2}(2m+1)x}{\sin \tfrac{1}{2}x}\, a_{m+1}\right]. \qquad (7)$$

We readily infer from conditions (A) and (B) that as m becomes infinite each term on the right-hand side of (7) approaches a limit uniformly in the interval $(0 < \delta \leqq x \leqq \pi)$; hence the same is true for the left-hand side. Moreover, we have from (7)

$$\int_0^\pi |\sigma_m(x)|\,dx \leqq \tfrac{1}{2}\left[\sum_{n=0}^{n=m-1}\left(\int_0^\pi \frac{\sin^2 \tfrac{1}{2}(n+1)x}{\sin^2 \tfrac{1}{2}x}\,dx\right)|\Delta^2 a_n| \right.$$

$$\left. + |\Delta a_m| \int_0^\pi \frac{\sin^2 \tfrac{1}{2}(m+1)x}{\sin^2 \tfrac{1}{2}x}\,dx + |a_{m+1}| \cdot \left|\int^\pi \frac{\sin \tfrac{1}{2}(2m+1)x}{\sin \tfrac{1}{2}x}\,dx\right|\right].$$

From conditions (A) and (B) and well-known properties of the integrals occurring on the right-hand side of (8) it follows that this side of the inequality remains less than a positive constant C for all values of m. Hence, for any $\delta > 0$, we have

$$\int_\delta^\pi |\sigma_n(x)|\,dx < \int_0^\pi |\sigma_n(x)|\,dx < C \quad (n = 1, 2, 3, \ldots),$$

and therefore

$$\int_\delta^\pi \left| \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx \right| dx = \lim_{n\to\infty} \int_\delta^\pi |\sigma_n(x)|\,dx \leqq C,$$

which proves the existence of $\int_0^\pi \left| \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx \right| dx$ and consequently establishes our theorem.

The criterion which we have just obtained is related to certain criteria due to W. H. Young and Szidon, but it includes cases of interest which do not come under these tests. Young's criterion may be stated as follows: (1) $\lim_{n\to\infty} a_n = 0$, (2) $\Delta^2 a_n \geqq 0$ $(n = 0, 1, 2, \ldots)$, where in case the first k of the a_n vanish, condition (2) need only be satisfied from $n = k$ on. For Szidon's criterion condition (2) is replaced by (2′) $\sum_{n=2}^{\infty} |\Delta a_n| \log n$ converges. The relationship between the three criteria may be illustrated by a few simple examples. For the sake of brevity we will symbolize Young's criterion by (Y), Szidon's by (S), and the criterion of the present paper by (M).

The coefficients of the series $\sum_{n=2}^{\infty} (\cos nx)/\log n$ satisfy (Y) or (M) but not (S). The coefficients of the series $\sum_{n=2}^{\infty} (\cos nx)/(\log n)^{1/2}$ satisfy (Y) but neither (S) nor (M). The set of constants defined by the relationships

$$a_{2n} - a_{2n+1} = [(n+1)\log^3(n+1)]^{-1},\ a_{2n+1} - a_{2n+2} = 0 \quad (n = 1, 2, 3, \ldots),$$

$$a_2 = (2\log^3 2)^{-1} + (3\log^3 3)^{-1} + (4\log^3 4)^{-1} + \ldots,$$

satisfy (S) but neither (Y) nor (M). The coefficients defined by

$$a_{2n} - a_{2n+1} = \frac{\sec(1/2n)}{2n\log^2 2n},\ a_{2n+1} - a_{2n+2} = \frac{\sec[1/(2n-1)]}{2n\log^2 2n} \quad (n = 1, 2, 3, \ldots),$$

$$a_2 = \frac{\sec 1/2}{2\log^2 2} + \frac{\sec 1}{2\log^2 2} + \frac{\sec 1/4}{4\log^2 4} + \frac{\sec 1/3}{4\log^2 4} + \ldots,$$

satisfy (M) but neither (Y) nor (S). The coefficients of the series $\sum_{n=1}^{\infty} (\cos nx)/n^\rho$ $(\rho > 0)$ satisfy (Y), (S) or (M).

[1] *Compt. rend.*, **192**, 144 (1931).

[2] Cf. Carslaw, *Theory of Fourier's Series and Integrals*, 3rd ed., §96, exs. 3 and 4.

[3] Young, *Proc. Lond. Math. Soc.*, [2], **12**, 41 (1912); Szidon, *Math. Zs.*, **10**, 121 (1921). Cf. also Tonelli, *Serie Trigonometriche*, §§93, 94; Hobson, *The Theory of Functions of a Real Variable and the Theory of Fourier's Series*, 2nd ed., vol. II, §403.

SPHERE GEOMETRY AND THE CONFORMAL GROUP IN FUNCTION SPACE

BY I. A. BARNETT AND DAVID NATHAN

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CINCINNATI

Communicated April 14, 1932

1. *Sphere Geometry in Function Space.*—By analogy with the definition of a sphere in Euclidean n-dimensional space,. a sphere in the Euclidean function-space R_x is represented by the equation[1]

$$\sigma\int f^2(y)dy - 2\int \varphi(y)f(y)dy + \omega = 0, \tag{1}$$

where $f(x)$ and $\varphi(x)$ are continuous functions on the interval $0 \leqq x \leqq 1$, and σ and ω are real numbers on the interval $0 \leqq \sigma, \omega \leqq 1$. Here, $\varphi(x)$, σ, ω are considered fixed for the moment, while $f(x)$ denotes the variable point in R_x. Writing (1) in the form

$$\int\left[f(y) - \frac{\varphi(y)}{\sigma}\right]^2 dy = \frac{\int\varphi^2(y)dy - \sigma\omega}{\sigma^2},$$

we observe that the distance from the fixed point $\varphi(x)/\sigma$ to the variable point $f(x)$ is constant. We shall call $\varphi(x)/\sigma$ the center of the sphere, while the radius r is given by $r^2 = \dfrac{\int\varphi^2(y)dy - \sigma\omega}{\sigma^2}$.

The angle θ between two spheres

$$\begin{aligned}\sigma_1\int f^2(y)dy - 2\int\varphi_1(y)f(y)dy + \omega_1 &= 0,\\ \sigma_2\int f^2(y)dy - 2\int\varphi_2(y)f(y)dy + \omega_2 &= 0,\end{aligned} \tag{2}$$

will be defined by the expression

$$\cos\theta = \frac{r_1^2 + r_2^2 - d^2}{2r_1r_2},$$

where r_1 and r_2 are the radii, and d the distance between the centers. Two spheres are said to intersect orthogonally when and only when $\cos\theta = 0$. It follows at once that a necessary and sufficient condition that two spheres (2) in R_x be orthogonal, is

$$2\int\varphi_1(y)\varphi_2(y)dy - \sigma_1\omega_2 - \sigma_2\omega_1 = 0. \tag{3}$$

We shall call the left member of (3) the *polar of the quadratic functional*

$$\int\varphi^2(y)dy - \sigma\omega.$$

A sphere (1) is completely fixed when $\varphi(x)$, σ and ω are given, so that we may take $\varphi(x)$, σ, ω as the homogeneous sphere coördinates of the

function-space R_x''. If then we take as element the sphere in R_x, we shall obtain a geometry which we shall call the sphere geometry of R_x since it is the analogue of the elementary sphere geometry of n-space. A second type of sphere geometry in R_x, analogous to Lie's sphere geometry in n-space, arises when use is made of the coördinates $\varphi(x)$, σ, ω, ρ connected by the relation

$$\rho^2 = \int \varphi^2(y)dy - \sigma\omega.$$

This type of geometry will be considered in a subsequent paper.

2. *The Group of Conformal Transformations.*—In order to proceed with the study of the sphere geometry in R_x we consider the linear, homogeneous functional transformations

$$\begin{cases} \tau\varphi'(x) = A(x)\varphi(x) + \int B(x, y)\varphi(y)dy + C(x)\sigma + D(x)\omega, \\ \tau\sigma' = \int E(y)\varphi(y)dy + F\sigma + G\omega, \\ \tau\omega' = \int H(y)\varphi(y)dy + K\sigma + L\omega, \end{cases} \tag{4}$$

which take a point $\{\varphi(x), \sigma, \omega\}$ of the function space R_x'' into another point $[\varphi'(x), \sigma', \omega']$. Here the functions $A(x)$, $B(x, y)$, $C(x)$, $D(x)$, $E(x)$, $H(x)$ are continuous on their respective ranges, while F, G, K and L are constants; τ is a factor of proportionality, so that $\{\varphi(x), \sigma, \omega\}$ and $\{\tau\varphi(x), \tau\sigma, \tau\omega\}$ are the same point in the function space R_x''.

Let us consider in particular the special transformations (4) which leave unchanged the quadratic functional equation

$$\int \varphi^2(y)dy - \sigma\omega = 0 \tag{5}$$

so that

$$\int \varphi'^2(y)dy - \sigma'\omega' = M^2[\int \varphi^2(y)dy - \sigma\omega],$$

where M is an arbitrary constant. The following relations are shown to exist between the coefficients of a transformation of this kind:

$$\begin{cases} \int C^2(y)dy - KF = 0, \int D^2(y)dy - GL = 0, A^2(x) - M^2 = 0 \\ 2\int C(y)D(y)dy - FL + M^2 - GK = 0, \\ 2A(x)C(x) - FH(x) - KE(x) + 2\int B(y, x)C(y)dy = 0, \\ 2A(x)D(x) - GH(x) - LE(x) + 2\int B(y, x)D(y)dy = 0, \\ 2A(x)B(x, y) - E(x)H(y) + 2\int B(z, x)B(z, y)dz + 2A(y)B(y, x) - \\ \quad E(y)H(x) = 0. \end{cases} \tag{4'}$$

Thus the transformations defined by (4) and (4') take a sphere of zero radius in into another sphere of the same kind. In other words, the most general linear point transformations in the space R_x'' which leave (5) invariant will be the analogue of the conformal transformations in n-space, expressed in homogeneous sphere coördinates. We shall therefore call the transformations (4) and (4') *conformal transformations in* R_x. It may be readily shown that such transformations are further characterized

by the property that they transform orthogonal spheres in R_x into orthogonal spheres.

We may show without difficulty that the product of two conformal transformations in R_x is another conformal transformation in R_x. The question of the existence of a unique inverse is determined by a method used by Hildebrandt[2] in the inversion of a projective transformation[3] in R_x. The system of linear integral equations (4) is reduced to a single Fredholm integral equation, whose Fredholm determinant

$$\begin{vmatrix} FG \\ KL \end{vmatrix} + \sum_{n=1}^{\infty} \int \cdots \int \begin{vmatrix} B(x_i, x_j)/A & C(x_i)/A & D(x_i)/A \\ E(x_j) & F & G \\ H(x_j) & K & L \end{vmatrix} \begin{matrix} dx_1 \ldots dx_2. \\ (i, j = 1 \ldots n) \end{matrix} \qquad (6)$$

will be called the *determinant of the conformal transformation.* It is seen at once that a conformal transformation (4) has a unique inverse if and only if the expression (6) is different from zero. It follows, further, that every non-singular conformal transformation of this kind, has a unique inverse and this inverse is again a non-singular conformal transformation in R_x.

Finally, since the determinant of the product of two conformal transformations in R_x is equal to the product of the separate Fredholm determinants of these transformations, it follows that the totality of non-singular transformations in R_x form a group which we shall call *the conformal group in* R_x.

3. *Infinitesimal Conformal Transformations.*—To obtain the most general infinitesimal conformal transformation in R_x we may, without loss of generality, take L in (4) to be unity, since this coefficient may be incorporated in the factor of proportionality τ. Thus we must determine the transformations of the form

$$\begin{cases} \dfrac{d\varphi(x)}{dt} = \alpha(x)\varphi(x) + \int \beta(x, y)\varphi(y)dy + \gamma(x)\sigma + \delta(x)\omega, \\ \dfrac{d\sigma}{dt} = \int \epsilon(y)\varphi(y)dy + a\sigma + b\omega, \\ \dfrac{d\omega}{dt} = \int \mu(y)\varphi(y)dy + c\sigma \end{cases}$$

which leave invariant the expression (5), and these are shown to be

$$\begin{cases} \dfrac{d\varphi(x)}{dt} = k\varphi(x) + \int \beta(x, y)\varphi(y)dy + \gamma(x)\sigma + \delta(x)\omega, \\ \dfrac{d\sigma}{dt} = 2\int \delta(y)\varphi(y)dy + 2k\sigma, \\ \dfrac{d\omega}{dt} = 2\int \gamma(y)\varphi(y)dy. \end{cases} \qquad (7)$$

where $\gamma(x)$ and $\delta(x)$ are arbitrary continuous functions, k an arbitrary constant and $\beta(x, y) + \beta(y, x) = 0$. The transformations (7) are precisely the regular infinitesimal conformal transformations obtained by Kowalewski[4] as the most general infinitesimal angle-preserving transformations in R_x. This again brings out the analogy with the situation in n-space.

Kowalewski showed that the infinitesimal conformal transformations in R_x written in non-homogeneous coördinates form a group in the sense that the commutator of any two such transformations is one of the same kind. Using a more general definition of commutator, we find that the transformations (7) constitute a group in the sense of Kowalewski.

A given infinitesimal conformal transformation in R_x generates a one-parameter group of non-singular conformal transformations in R_x, and there are formulas determining the coefficients of the generated finite transformations in terms of the coefficients of the infinitesimal transformations. The method by which these results are established is that used by Barnett[5] to obtain the corresponding results for the projective group in R_x.

[1] All integrations will be Riemannian and the range will be from 0 to 1.
[2] T. H. Hildebrandt, *Bull. Am. Math. Soc.*, **26**, p. 400 (1920).
[3] L. L. Dines, *Trans. Am. Math. Soc.*, **20**, p. 45 (1919).
[4] G. Kowalewski, *Compt. Rend.*, **153**, p. 1452 (1911).
[5] I. A. Barnett, *Bull. Am. Math. Soc.*, **36**, p. 273 (1930).

CONCERNING UNIORDERED SPACES

BY J. H. ROBERTS

DEPARTMENT OF MATHEMATICS, DUKE UNIVERSITY

Communicated April 7, 1932

The present paper is devoted to a solution of a problem proposed by G. T. Whyburn.[1]

In what follows the letter Z will denote a collection (or *system*) of closed point sets such that both the sum of every two elements of Z and every closed subset of an element of Z are elements of Z.

Definition.[2]—A separable metric space S is said to be *uniordered relative to a system* Z provided that for every point p of S there exists a monotone decreasing sequence of neighborhoods $U_1, U_2, U_3, \ldots$, of p whose boundaries $B_1, B_2, B_3, \ldots$ are elements of Z, and such that p is the only point common to all the sets $U_1 + B_1, U_2 + B_2, U_3 + B_3, \ldots$.

Problem.—Can every separable metric space S which is uniordered relative to a system Z be transformed by a biunivalued and continuous transformation into a separable metric space T in which every point p is contained in arbitrarily small neighborhoods with boundaries which are images of sets belonging to Z?

An example is given which shows the answer to the problem as stated to be in the negative. However, for the very important case where every element of Z is *compact* the answer is in the affirmative.

Example.—Let S denote the subset of the number plane consisting of the points A and B with coördinates $(0,1)$ and $(0, -1)$, respectively, and all points with coördinates (x,y) such that $x > 0$, $-1 \leqq y \leqq 1$. Let M denote the set of points such that $y = 0$, $0 < x \leqq 1$. Let L denote the collection of all intervals which are subsets of S, together with the point set M. Let L_1 denote the collection of all closed subsets of L. Finally let Z denote the collection of all sets each of which is the sum of a finite number of elements of L_1. Then every closed subset of an element of Z is an element of Z, and the sum of every two elements of Z is also an element of Z.

Now clearly S is uniordered relative to the system Z. It is to be noticed, however, that if W is an element of Z which separates A from B in S then there must exist a positive number e such that W contains all points of M for which $0 < x \leqq e$. Thus any two sets V_1 and V_2 which are elements of Z and which separate A from B in S must have a point in common.

Now suppose that there exists a biunivalued and continuous transformation L of S into a separable metric space T in which every point p can be enclosed in arbitrarily small neighborhoods whose boundaries are images of elements of Z. Then since $L(A)$ and $L(B)$ are distinct points it readily follows that there exist domains U_1 and U_2 containing $L(A)$ and $L(B)$, respectively, bounded by sets W_1 and W_2, respectively, and such that (1) $U_1 + W_1$ and $U_2 + W_2$ have no point in common, and (2) both W_1 and W_2 are images under L of elements of Z. Suppose D_1, D_2, V_1 and V_2 are the sets in S such that $L(D_i) = U_i$ and $L(V_i) = W_i (i = 1,2)$. Then D_1 and D_2 are domains containing A and B, respectively, and V_1 and V_2 are elements of Z which contain the boundaries of D_1 and D_2, respectively. Now V_1 and V_2 must be mutually exclusive since W_1 and W_2 are. But any two elements of Z which separate A from B in S must have a point in common. Now V_1 and V_2 separate A from B in S. Thus our supposition has led to a contradiction.

THEOREM: *Suppose the separable, metric space S is uniordered relative to a system Z of closed and compact point sets. Then there exists a biunivalued and continuous transformation of S into a separable metric space T in which every point p is contained in arbitrarily small neighborhoods whose boundaries are images of sets which belong to Z.*

Proof.—Every separable metric space is perfectly separable. That is, there exists a countable sequence $D_1, D_2, D_3, \ldots$ of domains such that every domain in S is the sum of one or more domains of this sequence. Let H and K be any mutually exclusive sets which are the sums of a finite number of domains of the sequence $D_1, D_2, D_3, \ldots$. Order all such pairs H, K into a countable sequence $H_1, K_1; H_2, K_2; H_3, K_3; \ldots$. If there exists a domain bounded by an element of Z, containing H_i but containing no point of K_i, then let R_i be such a domain. If no such domain exists then let R_i be the vacuous set. Now let T be the space containing identically the points of S but with the following definition of limit point: The point P is a limit point of the image of the set M if and only if the common part of every finite set of elements of the sequence $R_1, R_2, R_3, \ldots$ containing P also contains a point of M distinct from P.

Obviously the transformation thus defined is biunivalued. It is also continuous. For if P is a limit point of M (in S), then since for every i for which R_i contains P the set R_i is a domain, and since in S the common part of a finite set of domains is a domain, it follows that the common part of a finite set of domains R_i containing P also contains a point of M distinct from P, and therefore P is a limit point of the image of M in the space T.

It will now be shown that T is a regular[3] and perfectly separable space. Let D be any domain in T and let P be any point of D. Since P is not a limit point of T-D it follows, by definition of limit point in T, that there exist integers $n_1, n_2, \ldots, n_m$ such that each of the sets $R_{n_1}, R_{n_2}, \ldots, R_{n_m}$ contains P but their common part contains no point of T-D. The boundary in S of $R_{n_1} \cdot R_{n_2} \cdot \ldots \cdot R_{n_m}$ is a closed subset of the sum of a finite number of elements of Z, and is therefore itself an element of Z. Call this boundary B. Since by hypothesis B is compact it has the Borel property. As S is uniordered relative to Z there exists a monotonic sequence $U_1, U_2, \ldots$, of domains in S bounded by elements of Z, all containing P, and such that $P = \pi_1^{\infty} \bar{U}_i$. If Q is any point of B there exist integers i and n_i such that D_i contains Q and $D_i \cdot \bar{U}_{n_i} = 0$. Let $D_{j_1}, D_{j_2}, \ldots, D_{j_k}$ denote a finite set of domains covering B such that, letting w denote any one of the numbers $j_1, j_2, \ldots, j_k$, there exists an integer n_w such that $D_w \cdot \bar{U}_{n_w} = 0$. Let K denote $D_{j_1} + D_{j_2} + \ldots + D_{j_k}$. There exists an integer r such that $\bar{U}_r \cdot K = 0$. Let H be a set of the sequence $D_1, D_2, D_3, \ldots$ containing P and lying in U_r. The pair H, K is, for some integer i, the pair H_i, K_i. Then R_i is a domain containing H, and containing no point of K and hence no point of the boundary of the domain $R_{n_1} \cdot R_{n_2} \ldots R_{n_m}$. Then the common part of this domain and R_i contains P and lies with its boundary completely in D. Since the set of all domains obtainable in this way is countable it has been shown that T is both regular and perfectly separable. Hence it is metric.[4]

Let P be a point of T, and let e be any positive number. Let R denote the set of all points of T at a distance greater than $e/5$ from P. Then by definition of limit point there exists a domain X bounded by the image of an element of Z, containing P, and containing no point of R. Then the diameter of X is less than e. This completes the proof of the theorem.

Corollary.[5]—*Every separable, metric space S which is separated between each pair of its points can be transformed by a biunivalued and continuous transformation into a separable metric space which is zero-dimensional in the Menger-Urysohn sense.*

This corollary follows immediately from the theorem and the following lemma of Whyburn:

In order that a space be uniordered relative to a system Z it is necessary and sufficient that every two points of S may be separated by a set of the system Z.

[1] "Potentially Regular Point Sets," *Fund. Math.*, **16**, 160–172 (1930).

[2] Cf. Whyburn, loc. cit.

[3] Cf. Alexandroff and Urysohn, *Math. Ann.*, **92**, 263 (1924). A space is said to be *regular* provided that if P is a point of a domain D then there exists a domain E containing P and lying with its boundary in D.

[4] Cf. Chittenden, E. W., "The Metrization Problem," *Bull. Amer. Math. Soc.*, **33**, 13–34 (1927).

[5] Cf. Whyburn, loc. cit.

REMARKS ON A THEOREM OF D. V. WIDDER

BY J. KARAMATA

UNIVERSITY OF BEOGRAD, JUGOSLAVIA

Communicated March 30, 1932

In his note[1] entitled "On the Changes of Sign of the Derivatives of a Function Defined by a Laplace Integral," Widder sketches the demonstration of the following theorem:

THEOREM A. *Let*

$$f(x) = \int_0^\infty e^{-xt}\varphi(t)dt, \qquad \text{(I)}$$

$\varphi(t)$ being a continuous function for $t > 0$ and such that

$$\varphi(t) \text{ approaches a limit when } t \longrightarrow \infty. \qquad \text{(II)}$$

If the function $\varphi(t)$ changes sign at the points $0 < t_1 < t_2 < t_3 < \ldots < t_n$, without being identically zero in the neighborhood of these points, then the function $f^{(k)}(x)$ changes sign, for k sufficiently large, at the points $x_{1k} > x_{2k} > x_{3k} > \ldots > x_{nk}$ and

$$\lim_{k=\infty} \frac{x_{ik}}{k} = \frac{1}{t_i}, \; i = 1, 2, 3, \ldots n.$$

If the function $\varphi(t)$ has infinitely many changes of sign, the number of changes of sign of $f^{(k)}(x)$ also becomes infinite with k.

Widder's demonstration rests principally on the following fact:

When the function $\varphi(t)$ satisfies the condition (II) then

$$\lim_{k=\infty} \left[\frac{(-1)^k}{k!} x^{k+1} f^k(x) - \varphi\left(\frac{k}{x}\right) \right] =$$

$$\lim_{k=\infty} \left[\frac{1}{k!} \int_0^\infty e^{-t} t^k \varphi\left(\frac{t}{x}\right) dt - \varphi\left(\frac{k}{x}\right) \right] = 0 \quad \text{(III)}$$

uniformly for $0 \leqq x < \infty$.

For the proof of Theorem A the essential thing is that (III) should hold uniformly for x becoming infinite; that is, on setting $s = 1/x$, that

$$\lim_{k=\infty} \left[\frac{1}{k!} \int_0^\infty e^{-t} t^k \varphi(st) dt - \varphi(sk) \right] = 0 \qquad \text{(IV)}$$

uniformly for $s \leqq 0$.[2]

It is precisely this fact that permits one to see that the condition (II) is not essential, as Widder himself has remarked. Indeed, we can replace this condition by a more general one, and thus make the following changes in Theorem A:

THEOREM B. *Theorem A holds if we replace condition (II) by the following:*

$$\log |\varphi(t)| = O(t), \quad t \longrightarrow \infty, \qquad \text{(V)}$$

that is, there exists a number $M > 0$ such that

$$\lim_{t=\infty} e^{-Mt} \varphi(t) = 0. \qquad \text{(V')}$$

The proof of this theorem can be very simply reduced to that of Theorem A. Indeed, condition (V') being supposed satisfied, let

$$\Phi(t) = e^{-Mt} \varphi(t).$$

Then we have

$$F(x) = \int_0^\infty e^{-xt} \Phi(t) dt = \int_0^\infty e^{-(x+M)t} \varphi(t) dt = f(x + M).$$

Since $\Phi(t)$ approaches a definite limit when t becomes infinite (the limit zero), we may apply Theorem A to the functions $\Phi(t)$ and $F(x)$. The result of Theorem B follows immediately since the functions $\Phi(t)$ and $\varphi(t)$ have the same changes of sign,

$$F^{(k)}(x) = f^{(k)}(x + M) \text{ and } x \sim x + M.$$

Let us recall further that the existence of the integral (I) for every $x > 0$ does not imply the condition (V). To see this it is sufficient to consider the function

$$\varphi(t) = \exp t \cdot \exp \exp t \cdot \sin \exp \exp t$$

for which the integral (I) becomes, after the substitution $u = \exp \exp t$,

$$\int_e^\infty \left\{\log u\right\}^{-x} \sin u \, du,$$

and is consequently convergent for every $x > 0$. By taking into account the hypothesis that the integral (I) exists for every $x > 0$, (and by using theorems like those found in my note)[3] we can, however, replace condition (V) by a broader one: that there exist two numbers $M > 0$ and $m > 0$ such that

$$\left\{\varphi(t') - \varphi(t)\right\} > O(e^{Mt}), \quad t \to \infty$$

for all t' for which $t \leqq t' \leqq t + e^{-mt}$.

Finally we recall that Fekete[4] has given theorems of the same nature as A, but relative to integrals of the form $\int_0^1 f(t)(1-t)^k t^n dt$.

[1] *Proc. Nat. Acad. Sci.*, **18**, 112–114 (1932).

[2] The fact that (III) holds uniformly for $x \geqq 0$ is easy to verify since by virtue of (II), for $x \geqq 0$, $\alpha(t/x)$ approaches a constant uniformly (as t becomes infinite) and consequently (the integral in (III) or (IV) being a regular process of summation) $\frac{1}{k!}\int_0^\infty e^{-t} t^k \alpha(t/x)dt$ approaches the same constant uniformly for $x \geqq 0$, $k \longrightarrow \infty$.

[3] J. Karamata, *Compt. rend.*, **193**, 1156 (1931).

[4] M. Fekete, *Compt. rend.*, **190**, pp. 413 and 1366 (1930).